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# Plausibility arguments and universal gravitation

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**Abstract**

Newton's law of universal gravitation underpins our understanding of the dynamics of the Solar System and of a good portion of the observable universe. Generally, in the classroom or in textbooks, the law is presented initially in a qualitative way and at some point during the exposition its mathematical formulation is written on the blackboard and some quantitative consequences are discussed. In the present paper we argue that this approach can be improved by the use of plausibility arguments.

## 1. Introduction

Newton's law of universal gravitation underpins our understanding of the dynamics of the Solar System and of a good portion of the observable universe. At high school level, there are two well-tested ways of introducing this important law of nature to our students. The first way is to start with Newton's law of universal gravitation firstly by describing how it was discovered and how it works, and then by discussing some of its implications such as weight, weightlessness, the tides, and planet or/and satellite motion. This approach ends with a discussion on Kepler's three laws, see for example [1]. The other way is to follow the 'chronological order', that is starting with Kepler's laws of planetary motion as empirical laws, and proceeding to the discussion of Newton's law of universal gravitation [2]. In both approaches at some point after the qualitative introduction, the mathematical expression for the magnitude of Newton's law of universal gravitation is presented and its main features are discussed. Both procedures are standard and well tested, the present authors, however, believe that when introducing this fundamental law of nature, no matter the approach chosen by the teacher, arguing with appropriate plausibility examples may enhance the students's understanding of this important topic. An argument of plausibility is not a formal demonstration, but it may smooth the way to the acceptance of a theoretical result as a reasonable one though we still need to stress the necessity of its corroboration by experimental testing. Plausibility arguments in favour of Newton's law of universal gravitation can be found in a few university level textbooks, for example [2], but at the high school level they are harder to find. In what follows we review some plausibility examples and present new ones that can be useful when introducing Newton's law of universal gravitation in the classroom.

## 2. Using Kepler's third law

Consider the orbit of the Moon around the Earth. For the sake of simplicity let us suppose that the Earth is at rest with respect to the fixed stars and that we can consider the orbit of the Moon as a circle of radius $r$[1], see figure 1. Kepler's third

[1] The eccentricity of the lunar orbit is $\epsilon = 0.0549$. The eccentricity of the orbit of Mercury, the greatest in the Solar System after Pluto was demoted to the dwarf planet category, is $\epsilon = 0.2056$.

 

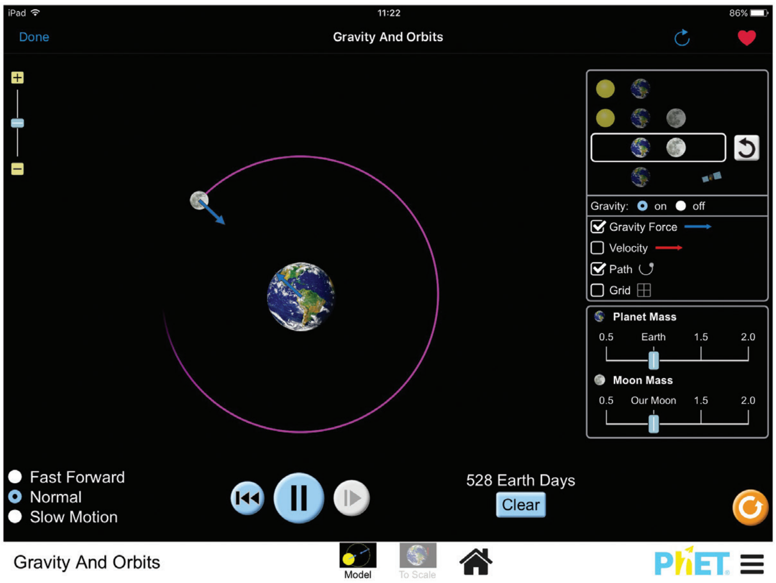


**Figure 1.** The Earth and the Moon in the recently launched PhET simulation *Gravity and Orbits*. Reproduced from [3], PhET Interactive Simulations, University of Colorado Boulder (https://phet.colorado.edu) made available under a CC BY 4.0 licence. Simulations such as this one assume that the student is familiar with Newtonian gravitation.

law applied to the system Earth–Moon tell us that the ratio of the square of the orbital period of the Moon $T$ to cube of the radius of its orbit is a constant

$$\frac{T^2}{r^3} = C. \tag{1}$$

If the orbit is a circle, the force that the Earth exerts on the Moon must be a centripetal one and given by

$$F = m\frac{v_0^2}{r}, \tag{2}$$

where $v_0 = 2\pi r/T$ is the orbital speed. Recall that in order to explain Kepler's second law (the law of areas) the gravitational force must be central; hence, for a circular orbit in which the centre of force and the geometrical centre coincide, the tangential or orbital speed must be constant. It follows that

$$F = \frac{4\pi^2 mr}{T^2}. \tag{3}$$

Eliminating $T$ with Kepler's third law we obtain

$$F = \frac{4\pi^2 m}{Cr^2}. \tag{4}$$

If we now invoke Newton's third law of motion then $\mathbf{F}_{\text{Moon–Earth}} = -\mathbf{F}_{\text{Earth–Moon}}$, and as a consequence the magnitudes of both forces are equal

$$\|\mathbf{F}_{\text{Moon}-\text{Earth}}\| = \|-\mathbf{F}_{\text{Earth}-\text{Moon}}\| = F, \tag{5}$$

where $F$ is given by equation (4). It follows that the constant $C$ must depend on the mass of the Earth $M$, and thus we write $1/C = 4\pi^2/GM$, where $G$ is the gravitational constant. Therefore,

$$F = \frac{GMm}{r^2}. \tag{6}$$

Notice that if we initially assume that $F \propto r^{-\nu}$, where $\nu$ is a real number, for a circular orbit Newton's second law of motion will read

$$\frac{C}{r^\nu} = \left(\frac{2\pi}{T}\right)^2 r, \tag{7}$$

or

$$\frac{C'}{r^{\nu+1}} = \frac{1}{T^2}. \tag{8}$$

Because Kepler's third law must be obeyed it follows that $\nu$ must be equal to 2; therefore $F \propto r^{-2}$. Writing $F = Cr^{-2}$, and using Newton's third law we can write $C = GMm$.

Once we have found the magnitude of the gravitational attraction we can discuss its vectorial features emphasizing in particular the role of Newton's third law. From this point on simulations such as the PhET group's simulation *Gravity and Orbits* [3]—see figure 1—or the more sophisticated one *Cavendish* [4] can be very useful. The approach discussed above follows closely the one discussed in [2] but there is an alternative simple way of arguing in favour of the plausibility of the law of universal gravitation at the high-school level.

## 3. The Moon and the apple

Let us consider once more the circular motion of the Moon around a fixed Earth, see figure 1. The orbital period of the Moon around the Earth is 27.3 d or $2.4 \times 10^6$ s and the radius of its orbit is $3.8 \times 10^8$ m, and consequently the Moon's centripetal acceleration is

$$a_c = \left(\frac{2\pi}{T}\right)^2 r \approx 2.6 \times 10^{-3}\ \mathrm{m\ s^{-2}}. \tag{9}$$

On the other hand, an apple (or any body) in free fall near the surface of the Earth has an acceleration $g$ approximately equal to 9.8 m s$^{-2}$, and the ratio $a_c/g$ is

$$\frac{a_c}{g} = \frac{2.6 \times^{-3}}{9.8} \approx 2.7 \times 10^{-4}. \tag{10}$$

Newton knew that the lunar orbital radius was more or less 60 times the radius of the Earth, that is $r = 60R$; therefore the square of ratio $R/r$ is

$$\frac{R^2}{r^2} = \frac{1}{3600} \approx 2.8 \times 10^{-4}. \tag{11}$$

This numerical quasi-coincidence is sufficiently alluring as to make us write

$$\frac{a_c}{g} = \frac{R^2}{r^2}, \tag{12}$$

or

$$a_c r^2 = gR^2 = C, \tag{13}$$

where $C$ is a constant. The force exerted on the Moon by the Earth and the lunar centripetal acceleration are linked by Newton's second law of motion, so we write

$$F = ma_c = mg\frac{R^2}{r^2}. \tag{14}$$

which with the help of equation (13) can be rewritten as

$$F = \frac{mC}{r^2}. \tag{15}$$

As before $F$ must obey Newton's third law, and this means that $C$ must be directly proportional to the mass of the Earth $M$. Writing $C = GM$, where $G$ is the universal constant we obtain the magnitude of universal gravitational attraction, equation (6).

There is another way of obtaining the result we want. Consider once more equation (14). In order to satisfy Newton's third law of motion the acceleration due to gravity $g$ must be directly proportional to the mass of the Earth that is

$$g \propto M. \tag{16}$$

On the other hand, if we consider the Earth as point mass then any information on its geometry must be suppressed; hence we write

$$g \propto M/R^2; \tag{17}$$

therefore we can also write

$$F \propto \frac{M}{\cancel{R^2}} \times \frac{m\cancel{R^2}}{r^2} \propto \frac{Mm}{r^2}. \tag{18}$$

In order to replace the proportionality symbol $\propto$ by an equality one $=$ we must insert a constant with the appropriate dimensions. This constant is the gravitational constant $G$; hence, once again equation (6) follows.

Let us take now another look at the 'Moon–apple' approach. According to John Conduitt (1688–1757), quoted in [5], the idea of a universal attraction occurred to Newton around 1666 during a sojourn at his mother's estate in Lincolnshire. Newton observing the fall of an apple asked himself if the gravitational attraction of the Earth would extend to the Moon influencing its motion. This story must be taken with a pinch of salt, see the final remarks, here we are concerned with its usefulness as an example of an argument of plausibility.

As before the idea is to compare the free fall of an apple near the surface of the Earth and the free fall of the Moon towards the centre of the Earth and from this comparison to infer the law of attraction between two massive bodies.

If we consider a time interval $\Delta t$ sufficiently short when compared to the orbital period of the Moon that is $\Delta t/T \ll 1$, we can make use of the kinematics of a projectile in a uniform gravitational field by considering the horizontal distance as the arc of circle of length $\Delta s$ in such a way that

$$\frac{\Delta t}{T} = \frac{\Delta s}{s}, \quad \Rightarrow \quad \Delta t = \frac{T}{s}\Delta s = \frac{\Delta s}{v_0}, \tag{19}$$

where as before $v_0 = s/T = 2\pi r/T$ is the magnitude of the orbital velocity.

If we consider the situation depicted in figure 2, the launching angle with respect to the horizontal line of reference is null, the vertical velocity component is also null but the horizontal component is equal to $v_0$. The initial height is equal to the radius of the orbit of the Moon around the Earth $r$, and thus the vertical fall of the Moon, $\Delta h_{\mathrm{Moon}}$, will be given by

$$\Delta h_{\mathrm{Moon}} = r - h_{\mathrm{Moon}} = \frac{a_{\mathrm{Moon}}}{2}\left(\frac{\Delta s}{v_0}\right)^2; \tag{20}$$

where $h_{\mathrm{Moon}}$ is the instantaneous height of the Moon and $a_{\mathrm{Moon}}$ is the magnitude of its centripetal acceleration. More details on the parabolic approximation to the orbit of the Moon can be found in the appendix. With the data at our disposal it follows that the magnitude of the centripetal acceleration is

$$a_{\mathrm{Moon}} = \left(\frac{2\pi}{T}\right)^2 r = \left(\frac{2\pi}{2.4\times10^6\ \mathrm{s}}\right)^2 \times (3.8\times10^8\ \mathrm{m}) \approx 2.6\times^{-3}\ \mathrm{m\ s^{-2}}. \tag{21}$$

In the same time interval, an apple in free fall near the surface of the Earth changes its height by

$$\Delta h_{\mathrm{apple}} = \frac{g}{2}\left(\frac{\Delta s}{v_0}\right)^2. \tag{22}$$

The ratio $\Delta h_{\mathrm{apple}}/\Delta h_{\mathrm{Moon}}$ is

$$\frac{\Delta h_{\mathrm{Moon}}}{\Delta h_{\mathrm{apple}}} = \frac{a_{\mathrm{Moon}}}{g} = \frac{2.6\times^{-3}}{9.8} \approx 2.7\times10^{-4}. \tag{23}$$

This ratio is almost equal to the ratio of the Earth's radius to the Moon's orbital radius

$$\frac{R^2}{r^2} = \frac{1}{3600} \approx 2.8\times10^{-4}. \tag{24}$$

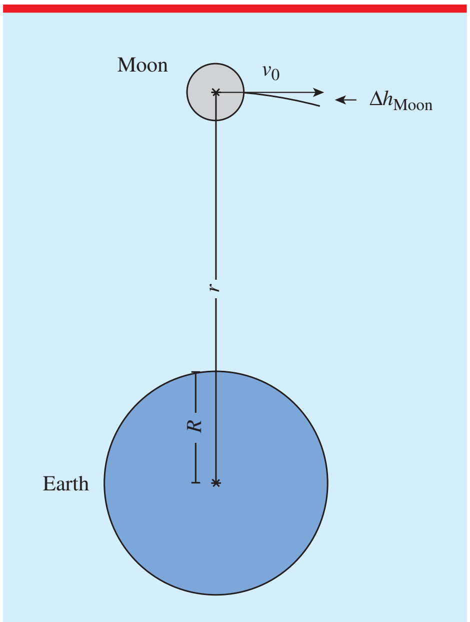


**Figure 2.** The Earth and the Moon: during the time interval $\Delta t \ll T$, the Moon falls vertically a distance $\Delta h_{\mathrm{Moon}}$ and moves horizontally a distance equal to $\Delta s = v_0\Delta t$.

How can we interpret this numerical coincidence? Firstly, it is a very strong indication that Newton's intuition was correct because it shows that is not implausible to extend the action of the Earth to the Moon. Secondly, it also indicates that this action follows a $r^{-2}$ power law. Suppose that the Moon is in free fall near the surface of the Earth, what is its acceleration? If the $r^{-2}$ power law holds then

$$\frac{a'_{\mathrm{Moon}}}{a_{\mathrm{Moon}}} = \frac{1/R^2}{1/r^2} = \frac{r^2}{R^2} = 3600, \tag{25}$$

where $a'_{\mathrm{Moon}}$ is the acceleration of the Moon near the Earth's surface. We have found that $a_{\mathrm{Moon}} = 2.6\times10^{-3}\ \mathrm{m\ s^{-2}}$; thus, it follows that

$$a'_{\mathrm{Moon}} = 3600\ a_{\mathrm{Moon}} = 3600\times2.6\times10^{-3}\ \mathrm{m\ s^{-2}} \approx 9.4\ \mathrm{m\ s^{-2}}. \tag{26}$$

Remarkable close to the mean value of $g$. Summing up we have plausible evidence that the force between the Earth and the Moon is proportional to $1/r^2$. Once again we can invoke Newton's third law to obtain the correct expression.

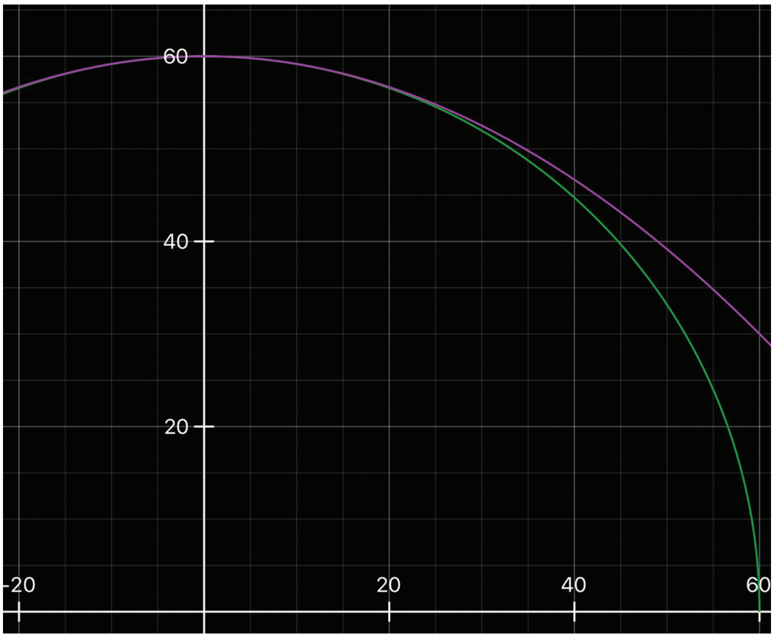


**Figure A1.** The green curve represents 1/4 of the circular orbit of the Moon: $y = \sqrt{60^2 - x^2}$ with $x \in [0, 60]$. The red curve represents the parabolic approximation $y \approx 60 - x^2/120$. For both axes the measure unit is the radius of the Earth $R$.

## 4. Final remarks

In this paper emphasis was given to the plausibility of the mathematical expression of the magnitude of the gravitational attraction. This must be completed by a discussion on its vectorial aspects. It may be also convenient to mention that the final result holds for elliptical orbits as well, and attention must be drawn to the universal character of the gravitational attraction, and finally, we must not forget to mention that the Earth, the Moon and the apple are treated as mass points.

The story of the apple must be mentioned with caution in the classroom. Though it cannot be entirely dismissed since there are reasons to believe that it must have played a role in Newton's idea of comparing the centrifugal force acting on the Moon to the terrestrial gravity, the story is also part of misconceptions about the genesis of the law of universal gravitation, in particular the belief that it was one in a sequence of brilliant ideas that Newton had in an extremely fruitful period of his life, the miraculous years of 1665–1666. [5]. It seems likely that the story was embellished and its main purpose was to be a corroboration of Newton's priority in the matter [6]. The reasoning that led Newton to the discovery of the law of universal gravitation is complex, see [5], chapter 5 in [7], or chapter 1 in Cohen and Whitman's translation into contemporary English of the *Principia* [8]. Anyway, the use of arguments of plausibility can smooth the way to a better understanding of Newton's law of universal gravitation by the beginner. Above all we should avoid introducing Newton's fundamental result on the gravitational attraction as an insight from the mind of a genius.

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## Appendix. The parabolic approximation

How good is the parabolic approximation? Consider one fourth of the circular orbit of the Moon as shown in figure A1. Its analytical representation reads

$$y(x) = +\sqrt{60^2 - x^2}, \quad x \in [0, 60],$$

where $x$ and $y$ are measured in units of the Earth radius. If we expand this equation in a Taylor series about $x = 0$ we will obtain

$$y(x) \approx 60 - \frac{x^2}{120} - \frac{x^4}{1728\,000} + \mathcal{O}(x^6).$$

If for small $x$ we consider only the first two terms in the Taylor expansion that is the parabolic approximation, the difference will be given by

$$\left| \sqrt{60^2 - x^2} - 60 + \frac{x^2}{120} \right| \approx \frac{x^4}{1728\,000}.$$

For $x = 1$, that is one Earth radius ($R = 6371$ km), the difference will be approximately equal to $5.8 \times 10^{-7}$. And if we set $x = 10$ the difference will be ten thousand times lesser but still very small, in fact it will be equal to $5.8 \times 10^{-3}$. This show us that the parabolic approximation is enough for our aim here.

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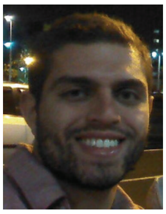

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