Work, gravitational energy, and the Great Pyramid

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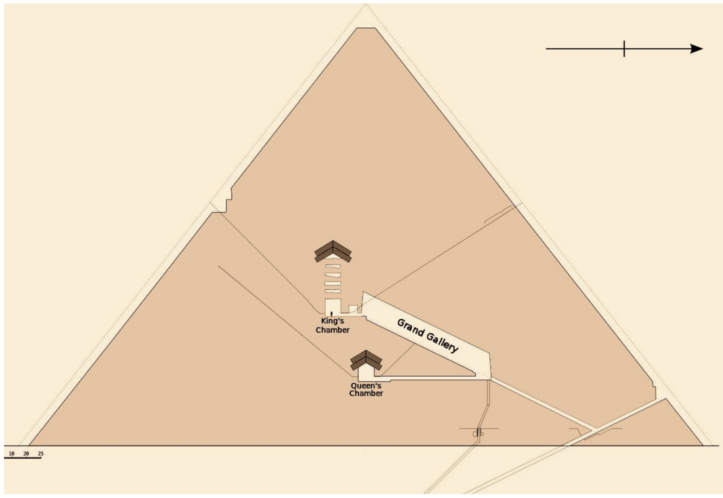

Figure 1. Internal sketch of the Great Pyramid. (Image Wikipedia.)

1. Introduction

The *Great Pyramid of Gizeh*, also known as *Khufu's Pyramid* or *Cheops' Pyramid* was built between 2560 b.C.E. and 2040 b.C.E. as a tomb for pharaoh Khufu of the IV dinasty [1]. Originally the Great Pyramid had a height of 146.7 meters and a square base whose side was 230.4 meters [1]. Ignoring the two mortuary rooms, the gallery, and the ascending and descending passages that correspond to small fraction of the total volume, see Figure 1, the Great Pyramid can be considered as an enormous block of solid rock with a mean density approximately equal to $2\,700\,\mathrm{kg/m^3}$. The Greek historian Herodotus (*c.* 484 BCE– *c.* 425 BCE) stated that it took 10 years to prepare the construction site and the subterranean rooms and another 20 years and a task force of 100 000 men‡ to build the pyramid properly speaking [2]. The problem was originally proposed in [3] where we can find numerical answers to compare with ours. Here, by making use of basic concepts of introductory mechanics, we discuss an analytical solution that allow us to check out Herodotus' statement. Numerical estimates are obtained and compared with [3] and more detailed calculations.

2. Work and gravitational energy

In order to find the work done to build the Great Pyramid we adopt the simplest point of view we can think of: We compute the mechanical work performed by an external agent which after completed will become stored as the gravitational potential energy of the pyramid with respect to the Earth. This simplification does not take into account many features involved in the process such as, for instance, the work done in quarrying and transporting the individual stones from the quarries to the construction site. Our point of view is akin to the one of child who is mounting a Lego kit with all the pieces already spread near her/him on the floor. In other words, we will compute the minimum amount of work employed to build the Great Pyramid.

‡ In fact, Herodotus writes 'Gangs totalling a hundred thousand labourers would work in rotation, three months at a time [2].

In a uniform gravitational field, we can replace all of the pyramid by a single point, the center of mass (c. of m.) which concentrates all of its mass. Taking into account the symmetry of the pyramid, we choose a Cartesian coordinate system such that z-axis is perpendicular to the base of the pyramid and goes through its geometrical center where we also place the origin of the coordinate system. By symmetry the c. of m. must be on the z-axis at the point $P(0, 0, z_{\text{c. of m.}})$, where $z_{\text{c. of m.}} > 0$ is the height of the c. of m.. With respect to the base of the pyramid we have

$$W_{\text{external agent}} = \Delta U_{\text{pyramid}} = M g\, z_{\text{c. of m.}}, \tag{1}$$

where M is the total mas of the pyramid, $g = 9,8$ m/s^2 is the gravitational acceleration near the surface of the Earth. Therefore, computing the work done against gravity is tantamount to evaluating $z_{\text{c. de m.}}$, and if we do this we will get the well-known result.

$$z_{\text{c. of m.}} = \frac{1}{4}\, H, \tag{2}$$

where H is the height of the pyramid. Therefore, the c. of m. is located at the point $(0, 0, H/4)$. The work done by the external agent reads

$$W_{\text{external agent}} = \frac{1}{4}\, M g\, H = \frac{1}{12}\, \rho\, g\, L^2\, H^2, \tag{3}$$

where L is the length of one side of the base and ρ is the mean density of the building material, mainly limestone ($2\,600$ kg/m^3) and granite ($2\,700$ kg/m^3 to $2\,800$ kg/m^3). This is the analytical solution we were looking for and all we have to do now is to plug in the available data. It then follows that

$$\begin{aligned} W_{\text{external agent}} &= \frac{1}{12} \times 2\,700 \times 9.8 \times (230.4)^2 \times (146.7)^2 \\ &\approx 2.5 \times 10^{12}\,\text{J}, \end{aligned} \tag{4}$$

where we have chosen a mean uniform density equal to $2\,700$ kg/m^3, and kept the estimated original dimensions of the Great Pyramid. Today the height is 138.8 meters.

3. Herodotus' estimation

Let us now check if Herodotus was right in claiming that it took 100 000 men and 20 years (2×10^6 man-years) to build the Great Pyramid. A modern young man (19-30 years-old) must burn more or less 2 500 kcal daily to keep healthy, 500 kcal/day more if moderately engaged in a physical activity in a regular way [4]. Also, a human being seen as a thermal machine has an estimated efficiency approximately equal to $20\,\%$ [4]. We cannot expect those numbers to be valid in the ancient Egypt, specially if we take into consideration the dietary and climate conditions. Therefore, following [3] let us suppose conservatively that in Ancient Egypt each worker consumed 1 500 kcal per day and that only 10 $\%$ of those calories were employed as useful lifting work. Our result in kcal (1 J $\approx 0,24$ cal) reads

$$W_{\text{lifting}} \approx 6.0 \times 10^8 \text{ kcal}. \tag{5}$$

One man is capable of performing 150 kcal of useful work per day, hence, one man-day is equivalent to 150 kcal. Suppose also that one working year is defined by 360 days. Then the work performed by one man in one year is equal to 360×150 kcal $= 5,4 \times 10^4$ kcal, that is, one man-year is equivalent to $5,4\,\times\,10^4$ kcal. Therefore, the work performed by N men in n years, let us denote it by W_{Nn}, is equal to

$$W_{Nn} = Nn\,\text{man-years}\,\times\,5.4\,\times\,10^4\ \text{kcal}. \tag{6}$$

This work is all employed in the building of the pyramid, that is, $W_{Nn} = W_{\text{external agent}}$. It follows that

$$Nn = \frac{6.0\,\times\,10^8}{5.4\,\times\,10^4} \approx 1.1\,\times\,10^4\ \ \text{man-years}, \tag{7}$$

Both results, i.e., the lifting work and the number of man-years are in agreement with [3], at least in order of magnitude. The result for the amount of lifting work also agrees with an estimation due to Wier, cited by Stewart in [5].

Our estimate is a hundred times lesser than Herodotus', but can be compared to modern estimates as the one by Illig and Löhner as cited in [6]. Illig and Löhner concluded that it took 6 700 workers and 10 years to build the Great Pyramid. The summary of their detailed estimation goes as follows

(i) 1170 workers in the quarries;

(ii) 1290 skippers on the River Nile;

(iii) 1020 workers on the land transportation of the stones;

(iv) 1320 workers in the construction site and at the foot of the pyramid;

(v) 880 workers on the pyramid flank;

(vi) 200 workers transporting the granite blocks;

(vii) 820 workers positioning the blocks on the pyramid plateau.

As mentioned before, our estimate does not take into account several aspects of the building process as we can see by examining the list above. We calculated only the work done against gravity and this could correspond to the 880 workers on the pyramid flank to whom we could add the 200 workers transporting the granite blocks in the Illig and Löhner's summary. There are more conservative estimates, for example, Lehner advocates 20 000 men in 20 years ($4\,\times\,10^5$ man-years) [7]. Comparison with Mendelssohn's results [8] is more difficult because it is nuanced and comprises the pyramid era. His numbers vary between 50 000 and 150 000 men in 100 years and takes into account the seasonality and the total work.

We can improve our estimation and still keep it simple enough so that it can be discussed at the high-school level if we take into account the mechanical energy losses. This can be done by supposing that the workers dragged a block of mass M equal to the total mass of the pyramid all along an inclined plane making an angle θ w.r.t. a horizontal straight line doing work against friction forces until the block is $H/4$ high above the ground level, see Figure 3. If μ is the sliding friction coefficient between the block and the inclined plane, the work done against the friction forces is seen to be given by

$$W_{\text{dragging}} = \frac{1}{4}\,MgH\mu\,\cot\theta = \mu\,\cot\theta\,W_{\text{lifting}} \tag{8}$$

and the total work by:

$$W_{\text{total}} = W_{\text{lifting}} + W_{\text{dragging}} = W_{\text{lifting}}\,\left(1+\mu\,\cot\,\theta\right), \tag{9}$$

Suppose we set $\mu = 0.75$, which is the sliding friction coefficient for limestone on limestone [10], and the inclined plane angle $\theta = 10^{\,0}$, then

$$W_{\text{dragging}} = 4.3\,W_{\text{lifting}}, \tag{10}$$

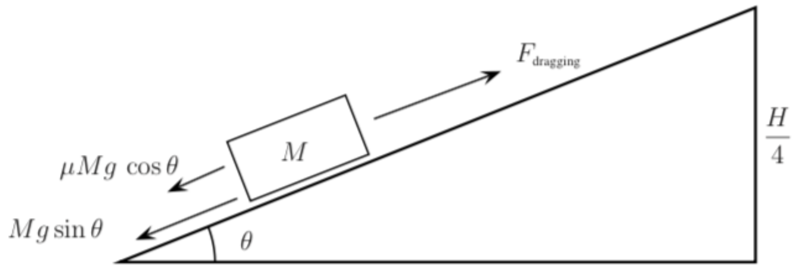

Figure 2. Dragging.

and total work is

$$W_{\text{total}} = 5.3\,W_{\text{lifting}} = 1.3\,\times\,10^{13}\,\text{J} = 3.1\,\times\,10^{9}\;\text{kcal}. \tag{11}$$

The number of man-years now is

$$Nn = \frac{3.1\,\times\,10^{9}}{5.4\,\times\,10^{4}} = 5.7\,\times\,10^{4}\;\;\text{man-years}, \tag{12}$$

that is, 5 700 men working during 10 years, still very far from Herodotus' claim but closer to Illig and Löhner's [6] if we add the last four items of their estimation totaling 3 220 men. We can do better if we decrease the inclined plane angle to, say $\theta = 5^{\,0}$, while keeping the same sliding coefficient. Then

$$W_{\text{dragging}} = 8.57\,W_{\text{lifting}}, \tag{13}$$

and

$$W_{\text{total}} = 9.6\,W_{\text{lifting}} = 5.7\,\times\,10^{9}\;\text{kcal}, \tag{14}$$

and hence

$$Nn = \frac{5.7\,\times\,10^{9}}{5.4\,\times\,10^{4}} = 1.1\,\times\,10^{5}\;\;\text{man-years}, \tag{15}$$

and increase of one order of magnitude in the number of man-years, or approximately eleven thousand men in a time span of ten years, or 5 500 men in 20 years. This lead us closer to the result due to Wier as quoted in [5] which is 10 000 men in 23 years, but one must keep in mind that Wier's estimation takes the time span for granted, and 23 years is the duration of Khufu's rule. Anyway, notice that as long as we keep the angle of inclination θ less than approximately $35^{\,0}$, for $\mu = 0.75$, we will have $W_{\text{dragging}} > W_{\text{lifting}}$.

The reader may find somewhat intriguing the way we have computed the work done by the sliding force after all this a non-conservative force and the work done depends on how we choose to go from one point to another. In the Appendix we perform this calculation in an alternative way.

An additional point that can also be discussed in the classroom is whether there was a surplus of energy to accomplish the task of building the Great Pyramid. The effort to answer this question may be large, but nevertheless we can try a back-of-the-envelope calculation with information easily available on the internet. The cultivated area provided by flooding of the river Nile is estimated to be between 20 000 and 34 000 square kilometers per year.

Pre-green revolution methods yielded 75 000 kg/km^2 of grains, and we consider their mean energy content to be approximately equal to 3 500 kcal/ kg. Then the annually available food energy is

$$2.4 \times 10^4 \text{ km/yr} \times 7.5 \times 10^4 \text{ kg/km}^2 \times 3.5 \times 10^4 \text{ kcal/kg} = 7.1 \times 10^{13} \text{ kcal/yr.}$$

Population in Ancient Egypt is estimated to be between 1.0 and 2.0 million of people. Let us set it at 1.5 million. Then the energetic intake per subject is

$$\frac{7.1 \times 10^{13} \text{ kcal/yr}}{1.5 \times 10^6 \text{ subjects}} = 4.7 \times 10^7 \text{ kcal/(yr} \cdot \text{subject)},$$

this equivalent to 1.3×10^5 kcal/(day·subject). Since we have estimated the energy intake of a worker to be 1.5×10^3 kcal/day, we see that this back-of-the-envelope estimation assures that there is enough energy to keep the workers and the rest of the population going on. Recall also that grains, wheat, barley, etc., were not the only sources of food, fish, figs, melons, vegetables were also consumed by the population.

4. Final remarks

To conclude we remark that we can further exploit the Great Pyramid estimations example by asking our students the carry out a similar analysis concerning work, gravitational energy, dietary requirements in the case of the building of the truncated pyramids of Central America and Mexico. Simple modifications of the calculations we have done here will suffice to provide answers. We can also try to make use of modern data on dietary requirements and verify that if we repeat this calculation plugging in modern data and increase the efficiency, it is possible to lower the numerical result for the work against gravity by one order of magnitude. We can also ask our students to compare results obtained to the ones concerning some modern buildings, e.g.: the Empire State Building (7 000 000 man-hours, one and a half years, height 381 meters). The example of the Great Pyramid, the only one of the Seven Wonders of the Ancient Word that has survived the flow of time, is a practical application of the concepts of mechanical work, gravitational potential energy. The historical setting, the estimations on the minimum dietary requirements and efficiency to accomplish the task give us a great opportunity to call the attention of our students to the importance of physics in branches of knowledge.

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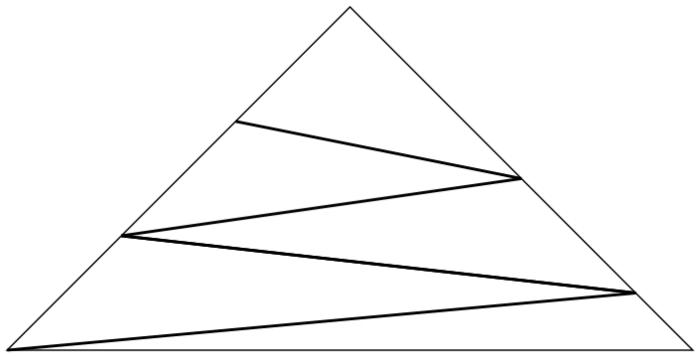

Figure 3. Constant inclination angle θ w.r.t. to an horizontal line zigzaging ramps.

Appendice: More on the dragging work

Consider an infinitesimal slab of rock of mass $dM = \rho\, A(z)dz$, where $A(z)$ is the cross section of the slab at the height z. In order to set this slab at height z we drag it along an imaginary inclined plane whose maximum height is H. This is equivalent to a set of zigzaging ramps all inclined at an angle θ w.r.t. the horizontal that serve the same purpose, see Figures 3 and 4. The work performed on this infinitesimal slab is

$$\begin{aligned} dW_{\text{dragging}} &= \mu dM g \cos\theta \, \frac{z}{\sin\theta} \\ &= \mu \cot\theta \, gz \, dM. \end{aligned}$$

The mass of the infinitesimal slab can be rewritten as

$$dM = \rho\, A(z) = \rho A_{\text{base}} \left(\frac{H - z}{H} \right)^2,$$

where $A_{\text{base}} = L^2$, and we have made use a theorem from the measure theory of volumes of cones and pyramids [11]. Therefore

$$W_{\text{dragging}} = \mu g \rho L^2 \cot\theta \frac{1}{H^2} \int_0^H (H - z)^2 \; z \, dz$$

Upon evaluating this integral and regrouping conveniently the individual terms we obtain

$$W_{\text{dragging}} = \mu \cot\theta \, W_{\text{lifting}},$$

which is the result we have worked with before.

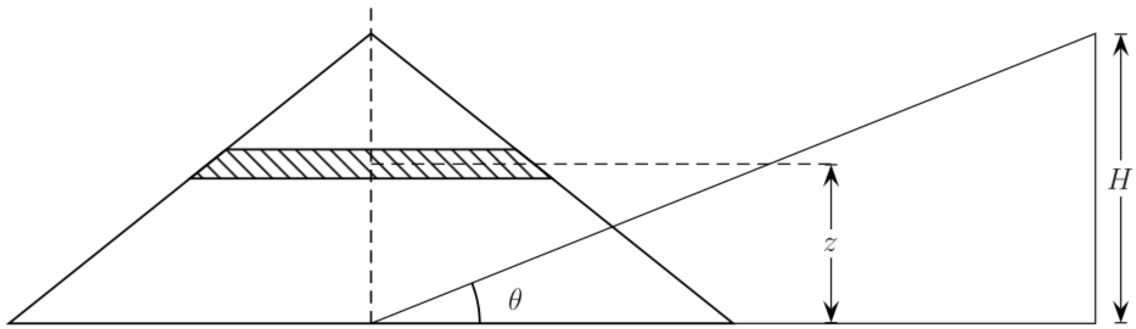

Figure 4. Zigzaging ramps can be replaced by a single ramp with an inclination angle θ.

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