Kepler's ellipse, Cassini's oval and the trajectory of planets

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Abstract

In this article, we discuss the similarities and differences between Kepler's ellipse and Cassini's oval with a small eccentricity. We show that these curves are barely distinguishable when the planetary orbits of our solar system are considered and that, from a numerical viewpoint, it is difficult to decide in favour of one of them.

Keywords: Kepler's ellipse, Cassini's oval, orbits

(Some figures may appear in colour only in the online journal)

1. Introduction

It is well known that Johannes Kepler was a key figure in the 17th century scientific revolution and he played an important role in the search for a better description of planetary motion. In his book, *Astronomia Nova*, which was published in 1609 [1], he provided strong arguments for heliocentrism and the elliptical trajectory of the planets around the Sun. However, as pointed out by Cohen [2], Kepler was not immediately believed. In fact, in 1675, Giovanni Domenico Cassini, the French Royal Astronomer, did not agree with Kepler and he tried to prove that the planetary orbits were ovals [3].

More recently, Sivardiere [4] explored this question and concluded that the difference between the ellipse and Cassini's oval is as distinguishable as that between the ellipse and the circle; therefore, if we discard the circle in favour of the ellipse, then, we also should discard the oval with the same argument. In this article, we analyse this possibility by drawing attention to the similarities and differences of these two curves for small eccentricities, and show that, based only on numerical calculations, it is difficult to decide in favour of one of the two curves.

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The most common misconception found in undergraduate non-major students from different countries, as indicated in [5, 6], is the belief that planetary orbits around the Sun are highly elliptical. A less common but equally false idea is that planetary orbits include a mix of circular and highly elliptical orbital shapes. Astronomy-education researchers have been working to solve these and other related problems, and new learning strategies have been discussed in [7, 8]. The discussion presented in this paper offers a context to address these conceptual and reasoning difficulties, and should be useful for students and instructors of general education astronomy and classical mechanics courses at the undergraduate level.

2. Analysis of the curves

Let us initially consider an ellipse whose centre is at the origin of the coordinates, and whose major semi-axis, a, and minor semi-axis, b_k, coincide with the coordinates x and y, respectively. The foci, f_1 and f_2, are localised at $-c$ and c with respect to the origin of the coordinate system on the horizontal axis, x. Its polar equation is then

$$r_k^2 = \frac{b_k^2}{1 - \epsilon^2 \cos^2 \theta}, \tag{1}$$

where the eccentricity ϵ is

$$\epsilon = \frac{c}{a}, \tag{2}$$

and the angle θ is the angle between the radius r_k and the horizontal axis. Therefore, when $\theta = 0$, we have $r_k = a$, and then (1) gives

$$a^2 = \frac{b_k^2}{1 - \epsilon^2}. \tag{3}$$

After solving (3) for the minor semi-axis, b_k, we find that it is related to the major semi-axis a and the focal distance c as

$$b_k = \sqrt{a^2 - c^2}. \tag{4}$$

A Cassinian oval is the locus of a point P such that the product of its distances, r_1 and r_2, from the two fixed points, f_1 and f_2, is a constant equal to d^2. Let us also consider that the major semi-axis of this oval is equal to the semi-major axis of the ellipse, its centre is also at the origin of coordinates, and its semi-axes coincident with the coordinates. Its polar equation is then

$$r_c^4 + c^4 - 2r_c^2 c^2 \cos(2\theta) = d^4. \tag{5}$$

At $\theta = 0$, we have $r_c = a$, and (5) is reduced to

$$a^4 + c^4 - 2a^2 c^2 = d^4. \tag{6}$$

After solving (6) for d, we obtain

$$d = \sqrt{a^2 - c^2}. \tag{7}$$

At $\theta = \pi/2$, we have $r_c = b_c$, and (5) is recast to

$$b_c^4 + c^4 + 2b_c^2 c^2 = d^4. \tag{8}$$

After solving (8) for b_c, we get

$$b_c = \sqrt{d^2 - c^2}. \tag{9}$$

Equations (4) and (7) show that, as far as the ellipse and the oval have the same major semi-axis and same eccentricity, the constant d in the oval equation and the minor semi-axis b_k of the ellipse have the same value:

$$d = b_k. \tag{10}$$

3. Analysis of the curves for a small eccentricity

When the eccentricity is small, $c \approx 0$, the polar equation (5) for the oval can be shortened to

$$r_c^4 - 2r_c^2 c^2 \cos(2\theta) \approx d^4. \tag{11}$$

In a first approximation, $r_c \sim d$ and (11) is rewritten as

$$r_c^4 \left[1 - 2 \left(\frac{c}{d} \right)^2 \cos(2\theta) \right] \approx d^4. \tag{12}$$

Solving (12) for r_c^2 we obtain

$$r_c^2 \approx \frac{d^2}{1 - (c/d)^2 \cos(2\theta)}. \tag{13}$$

After replacing the trigonometric identity $\cos(2\theta) = 2\cos^2\theta - 1$ into (13), we get

$$r_c^2 \approx \frac{d^2}{1 + (c/d)^2 - 2(c/d)^2 \cos^2\theta}. \tag{14}$$

Therefore, with the help of (10), given the minor semi-axis, b_k, and eccentricity $\epsilon = c/a \approx 0$ of the ellipse, the polar equation of the Cassinian oval with the same major semi-axis and eccentricity is

$$r_c^2 \approx \frac{b_k^2}{1 + (c/b_k)^2 - 2(c/b_k)^2 \cos^2\theta} \tag{15}$$

and, at this approximation, (15) is very similar to (1). In fact, the difference between these two radii is

$$r_k - r_c \approx \frac{c^2}{b_k} \left[\frac{1}{2} - \left(1 - \frac{b_k^2}{2a^2} \right) \cos^2\theta \right]. \tag{16}$$

Therefore, at $\theta = 0$, the difference between these radii, $r_k = a$ and $r_c = a'$, where a' is the major semi-axis given by the approximation (15), is

$$\frac{a - a'}{a} \approx -\frac{c^2}{2ab_k} \left(1 - \frac{b_k^2}{a^2} \right), \tag{17}$$

and, hence, we conclude that a' is always greater than a. For small eccentricities we also have that

$$b_k \approx a \left(1 - \frac{1}{2}\epsilon^2 \right), \tag{18}$$

and then, we can rewrite (17) as follows:

$$\frac{a - a'}{a} \approx -\frac{1}{2}\epsilon^4 \left(1 + \frac{1}{2}\epsilon^2 \right). \tag{19}$$

Accordingly, at $\theta = \pi/2$, (16) also shows that we have the maximum difference between these two radii for $r_k = b_k$ and $r_c = b_c'$, where b_c' is the minor semi-axis for the oval given by the approximation (15):

$$\frac{b_k - b_c'}{b_k} \approx \frac{c^2}{2b_k^2}. \tag{20}$$

Again, by using the approximation (18), the right-hand side of (20) is recast into the form

$$\frac{b_k - b_c'}{b_k} \approx \frac{1}{2}\epsilon^2(1 + \epsilon^2) \tag{21}$$

and we can also conclude that b_c' is always lower than b_k.

Table 1. Relative values for the minor semi-axis of Kepler's ellipse, b_k, and the minor semi-axis of Cassini's oval, b_c, for the planets of the solar system. The signal ($'$) corresponds to the approximated values given by equations (19) and (21), for the major semi-axis, a', and the minor semi-axis, b'_c, respectively.

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
a	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
ϵ	0.205 600	0.006 700	0.016 700	0.093 500	0.048 900	0.05650	0.045 700	0.011 300
equation (2) c	0.205 600	0.006 700	0.016 700	0.093 500	0.048 900	0.05650	0.045 700	0.011 300
equation (4) b_k	0.978 636	0.999 978	0.999 861	0.995 619	0.998 804	0.998 403	0.998 955	0.999 936
equation (7) d	0.978 636	0.999 978	0.999 861	0.995 619	0.998 804	0.998 403	0.998 955	0.999 936
equation (9) b_c	0.956 795	0.999 955	0.999 721	0.991 219	0.997 606	0.996 803	0.997 909	0.999 872
$(b_k - a)/b_k$	−0.021 830	−0.000 022	−0.000 139	−0.004 400	−0.001 198	−0.001 600	−0.001 046	−0.000 064
$(b_k - b_c)/b_k$	0.022 318	0.000 022	0.000 139	0.004 419	0.001 199	0.001 603	0.001 047	0.000 064
$(b_k - b'_c)/b_k$	0.022 029	0.000 022	0.000 139	0.004 409	0.001 198	0.001 601	0.001 046	0.000 064
$(a - a')/a$	−0.000 912	0.000 000	0.000 000	−0.000 038	−0.000 003	−0.000 005	−0.000 002	0.000 000

If the eccentricity for each planet, ϵ, is supposed to be known (its numerical value can be found in [9]), then we chose the major semi-axis to be equal to the unit and note that the foci's position, $-c$ and c, the minor semi-axis for the ellipse, b_k, the square root of the product of the oval's radii, d, and the minor semi-axis for this oval, b_c, for the Cassinian oval are determined with the help of (2), (4), (7) and (9), respectively. With all the parameters for these two curves well defined, we then determined the relative values for the difference between the minor semi-axis of the ellipse, b_k, and the circle of radius a. We have also calculated the relative values for the difference between the minor semi-axis for the ellipse, b_k, and compared it with the same axis for the oval, b_c. These calculations are summarised in table 1.

Table 1 also shows that the maximum difference between the positions given by the circumference and the ellipse is of the same order of magnitude as that for the maximum difference between the positions given by the Cassinian oval and the ellipse, which is in agreement with Sivardiere's conclusion: '[...] the difference between an oval and an ellipse having a common focus and a common axis a is of the same order as the difference between the ellipse and a circle of radius a'. However, this difference is too small when compared with the values of the axis for all these curves to be observed with the naked eye, as was the case in Kepler's day [10]. From our results, we remark that, for small eccentricities, the differences for the major semi-axis given by this approximation, a', (19), and the minor semi-axis, b'_c, (21), are very small. We can also note that a' is greater than a, and b'_c is greater than b_c, only if we consider the eccentricities with six decimal places and measure them, in mean, until the sixth decimal place and the fifth decimal place, respectively.

Wilson [11] calls our attention to the fact that Kepler was probably unable to geometrically observe the difference between the ellipse and the circle unambiguously. By the same token, we can suppose that the difference between the ellipse and the oval is also indistinguishable for the standards of this period (see also [12]). Indeed, we can better appreciate these arguments through figures 1 and 2, where a blue ellipse defined by (1) and a red oval defined by (15) are plotted by using the eccentricity of Mars and Mercury, respectively. The corresponding ellipse and oval for Mars, shown in figure 1, are barely indistinguishable. In figure 2 we illustrate both trajectories with the eccentricity for Mercury by using the same equations (1) and (15). In this case, the curves are distinguishable only at positions near the minor semi-axis. We recall that seven of the eight planets have an eccentricity that is lower than that for Mars, with the only exception being Mercury, the closest planet to the Sun and also one of the most difficult to observe.
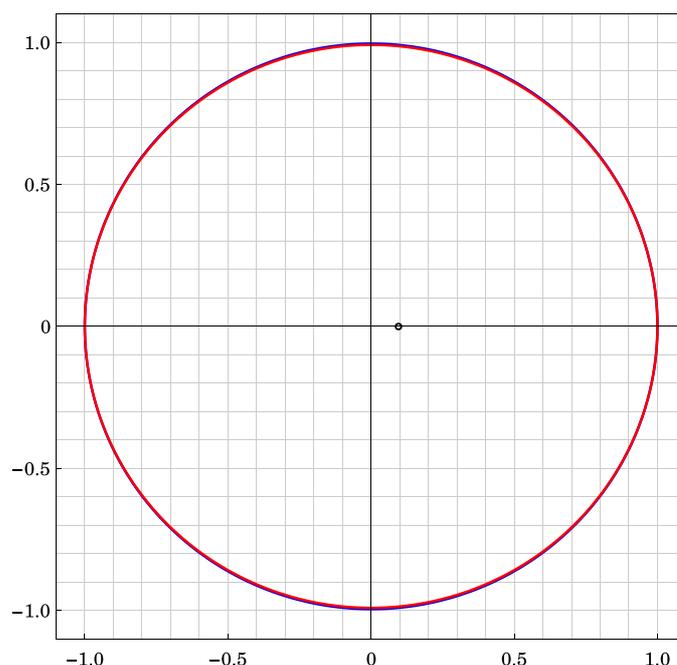
Figure 1. A blue outer Kepler's ellipse and a red inner Cassinian oval, as defined by (1) and (15), plotted with Mars's parameters: major semi-axis $a = 1.000\,000$, minor semi-axis for the ellipse, $b_k = 0.995\,616$, and eccentricity, $\epsilon = 0.093\,500$. At this small eccentricity they are barely distinguishable. We also remark that their maximum difference is on the minor semi-axis. One of the foci is also indicated at $(c, 0)$ with $c = 0.093\,500$.

4. Final remarks

From the results presented in table 1 and illustrated in figures 1 and 2, it is not surprising that many of the researchers who have followed Kepler have also tried to understand and describe the planetary motions around the Sun. Kepler's ellipse and Cassini's oval are barely distinguishable when orbits with a small eccentricity are considered. Their observational measurements give the possibility to conjecture that these two curves can describe the trajectory of the planets of our solar system. Moreover, our discussion also shows that to discard the circle and establish the ellipse as the shape of a planet's orbit is not as straightforward a geometric affair as it is generally assumed to be in higher secondary and undergraduate courses. This illustrates the ingenuity of Kepler in analysing the observational data at his disposal.

Finally, we call attention to Laplace's remark, found in his 'Mécanique Céleste', that only with Newton and his gravitation law will the ellipse be elected as the curve to better describe planetary motions, and all incompatibilities between theory and the real orbit are caused by the disturbance of another celestial body [13]. In Laplace's words: 'we have shown that this law follows the inverse ratio of the square of distances. It is true, that this ratio was deduced from the supposition of a perfect elliptical motion, which does not rigorously accord with the observed motions of the heavenly bodies'.
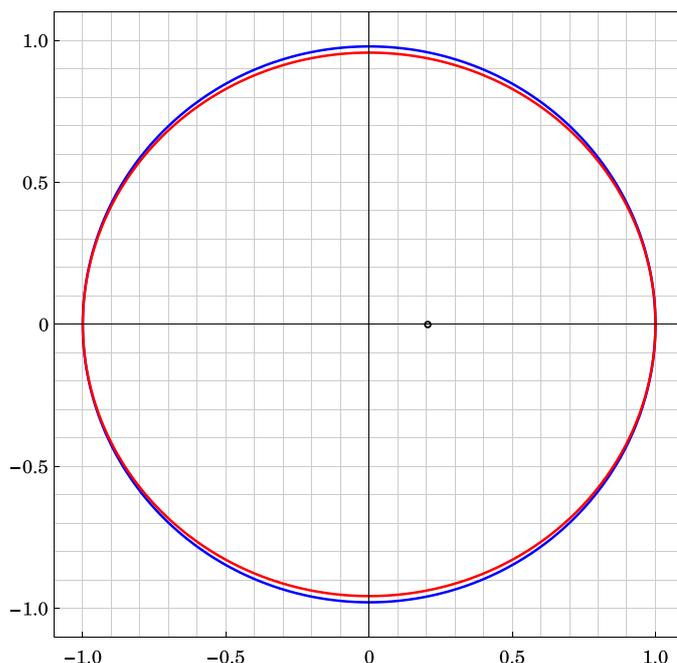
Figure 2. A blue outer Kepler's ellipse and a red inner Cassinian oval, as defined by (1) and (15), plotted with Mercury's parameters: major semi-axis $a = 1.000\,000$, minor semi-axis for the ellipse $b_k = 0.978\,636$ and eccentricity, $\epsilon = 0.205\,600$. With eccentricity values as high as 0.2 they are distinguishable only at positions near to the minor semi-axis. One of the foci is also indicated at $(c, 0)$ with $c = 0.205\,600$.

Acknowledgment

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