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How to introduce the magnetic dipole moment

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Abstract
We show how the concept of the magnetic dipole moment can be introduced in the same way as the concept of the electric dipole moment in introductory courses on electromagnetism. Considering a localized steady current distribution, we make a Taylor expansion directly in the Biot–Savart law to obtain, explicitly, the dominant contribution of the magnetic field at distant points, identifying the magnetic dipole moment of the distribution. We also present a simple but general demonstration of the torque exerted by a uniform magnetic field on a current loop of general form, not necessarily planar. For pedagogical reasons we start by reviewing briefly the concept of the electric dipole moment.

1. Introduction

The general concepts of electric and magnetic dipole moments are commonly found in our daily life. For instance, it is not rare to refer to polar molecules as those possessing a permanent electric dipole moment. Concerning magnetic dipole moments, it is difficult to find someone who has never heard about magnetic resonance imaging (or has never had such an examination). In this examination the patient is subjected to an external magnetic field approximately 30 000 times greater than the terrestrial magnetic field and a crucial point is that the nucleus of each hydrogen atom of the patient's tissue to be analysed has a magnetic dipole moment that interacts in some way with the external field. These are just two, among a great number of examples that could be given.

The electric dipole moment can be viewed as the source of a peculiar electric field, properly called the field of an electric dipole, or as the distinctive property of an electric system which behaves peculiarly under the influence of an external electric field. The electric dipole field is obtained as the leading contribution in a Taylor expansion of the field of a localized static charge distribution with zero total charge at distant points. The behaviour of the dipole in an external electric field is the tendency of the dipole moment to align with the external field direction due to the torque from this field. In both views exactly the same quantity is obtained and defined as the electric dipole moment of the electrical distribution. It is important to note that a magnetic dipole moment is perfectly analogous to the electric dipole moment: it is the source of a magnetic field obtained as the leading contribution in a Taylor expansion of the field of a localized steady current distribution at distant points and also has a tendency to align to an external magnetic field. Actually, the electric field of the electric dipole and the magnetic field of the magnetic dipole have practically identical expressions; the same occurs for the expressions of the torque on the electric dipole in an external electric field and of the torque on the magnetic dipole in an external magnetic field. Obviously, electric and magnetic dipole moments are quite distinct physical quantities, but to define one of them as the source of a peculiar field and the other as responsible for a peculiar behaviour in the presence of an external field is to introduce an artificial asymmetry between the two systems which possibly conveys a false impression to the student. Therefore, it is pedagogically preferable to show the perfect analogy of the two concepts by introducing them in the same way to the students.

However, in most of the textbooks we have consulted, only the electric dipole moment is introduced with the aid of a Taylor expansion, but not the magnetic dipole moment. Indeed, most introductory textbooks [1–10] introduce the magnetic dipole moment in a quite unnatural way. Without any further consideration, they compute the torque exerted by a constant and uniform magnetic field on a current loop and compare the result with the torque exerted by a constant and uniform electric field on an electric dipole. Exceptions to this procedure can be found in other textbooks [11, 12], where a Taylor expansion for the magnetic field of a particular current loop at particular points is used for the identification of the magnetic dipole moment of the current loop, and in other textbooks [13, 14], where the magnetic dipole moment is introduced when magnetism in matter is discussed. It is worth mentioning that in [15], the author defines both electric and magnetic dipole moments in terms of torques in external fields.

Our main purpose here is to propose a natural way of introducing the concept of the magnetic dipole moment in the same spirit as [11, 12], but in a much more general way, since these authors based their discussions on the very particular case of a circular current loop and considered points only on its symmetry axis. The idea we pursue here matches exactly that commonly used to introduce the electric dipole moment. Our second purpose is to present a simple but general calculation of the torque exerted on a current loop of arbitrary form by an external, uniform magnetic field, since all textbooks consulted by the authors [1–12] compute this torque in very particular cases, namely, by choosing planar current loops (rectangular ones or circular ones, at most) in the presence of external uniform magnetic fields with very particular orientations relative to the plane of the current loop.

This paper is organized as follows. In section 2 we review how the electric dipole moment arises naturally from an appropriate Taylor expansion. In section 3, we propose how to introduce the magnetic dipole moment in a way analogous to that of section 2. In section 4 we present an original calculation for the torque exerted by an external uniform magnetic field on a generic current loop, not necessarily planar, with arbitrary orientation relative to the magnetic field. Section 5 is left for the final remarks.

2. Electric dipole moment: a brief survey

This section was introduced for pedagogical reasons. As well as noting how the concept of the electric dipole moment of a localized charge distribution arises when we look for an approximate expression for the electric field at a point very far removed from the distribution, our purpose here is also to establish notation and basic ideas. For simplicity, but without

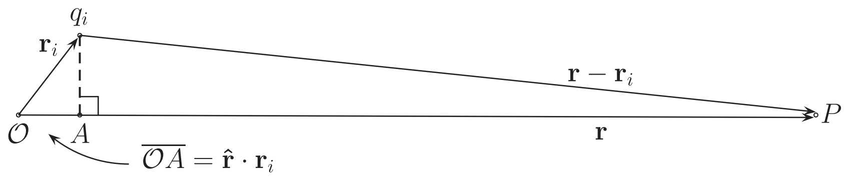

Figure 1. Origin $\mathcal{O}$, the vector position $\mathbf{r}_i$ of a generic charge q_i of the system, a point P very far from the distribution, where the electric field is to be evaluated, and the projection of $\mathbf{r}_i$ on the direction of $\hat{\mathbf{r}}$, namely, $\overline{\mathcal{O}A} = \hat{\mathbf{r}}\cdot\mathbf{r}_i$.

any loss of generality, instead of considering a charge distribution described by a volumetric charge density ρ, we shall consider a charge distribution composed by N point charges, q_i, $i = 1, 2, \ldots, N$, fixed at positions $\mathbf{r}_i$ ($i = 1, 2, \ldots, N$), respectively. The exact expression for the electrostatic field created by this system at any point of space, except at the positions of the charges, is given directly by Coulomb's law and the superposition principle, namely,

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0}\frac{(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|^3}\,. \tag{1}$$

For many purposes, for a point very far from the distribution it suffices to work with an approximate expression for the electric field $\mathbf{E}(\mathbf{r})$. By 'very far' we mean that $r_i \ll r$ ($i = 1, 2, \ldots, N$), where $r_i = |\mathbf{r}_i|$ and $r = |\mathbf{r}|$. Then, since $r_i/r \ll 1$, we may use the following approximation

$$|\mathbf{r}-\mathbf{r}_i| \approx \overline{AP} = r - \hat{\mathbf{r}}\cdot\mathbf{r}_i\,, \tag{2}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$. This approximation can be interpreted geometrically with the aid of figure 1, where points $\mathcal{O}$ (the origin), A and P are shown.

Equation (2) and approximation $(1+x)^n \approx 1+nx$, valid for $|x| \ll 1$, can be used to yield an approximate expression for the rhs of equation (1). With this goal, note that

$$\frac{1}{|\mathbf{r}-\mathbf{r}_i|^3} \approx \frac{1}{(r-\hat{\mathbf{r}}\cdot\mathbf{r}_i)^3} = \frac{1}{r^3}\left(1-\frac{\hat{\mathbf{r}}\cdot\mathbf{r}_i}{r}\right)^{-3} \approx \frac{1}{r^3} + \frac{3(\mathbf{r}_i\cdot\hat{\mathbf{r}})}{r^4}\,. \tag{3}$$

Substituting (3) into (1) and maintaining terms only up to linear order in r_i, we obtain

$$\mathbf{E}(\mathbf{r}) \approx \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2} + \frac{1}{4\pi\epsilon_0}\left[\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}}{r^3}\right], \tag{4}$$

where we defined

$$q = \sum_{i=1}^{N} q_i \qquad \text{and} \qquad \mathbf{p} = \sum_{i=1}^{N} q_i\mathbf{r}_i\,. \tag{5}$$

The quantity q is readily identified as the total charge of the distribution, while the vectorial quantity $\mathbf{p}$ is defined as the electric dipole moment of the distribution. As expected, for very large distances from the distribution, the electric field of the system can be considered, in a first approximation, as that of a point charge q fixed at the origin. However, for a neutral system ($q = 0$) the first approximation for the electric field is given by the second term on the rhs of equation (4). This term is the dipole contribution for the electric field of the system.

In the particular case where the system is composed only of two opposite charges, q_0 and $-q_0$ ($q_0 > 0$), localized respectively at positions $\mathbf{r}_+$ e $\mathbf{r}_-$, the electric dipole moment of the system written in (5) takes the simple form $\mathbf{p} = q_0(\mathbf{r}_+ - \mathbf{r}_-) = q_0\mathbf{d}$, a common result found in any introductory textbook on electromagnetism. The expression $\mathbf{p} = q_0\mathbf{d}$, $\mathbf{d}$ being the vector from the negative charge to the positive one, is then the electric dipole moment of this simple system. It is usual in the literature to call a system formed by two opposite charges $\pm q_0$ by the name of an electric dipole (two electric and opposite poles). Had we started with a continuous charge distribution described by a volumetric charge density ρ in a region $\mathcal{R}$, we would have obtained for the electric dipole moment of the system the expression $\mathbf{p} = \int_{\mathcal{R}} \rho(\mathbf{r})\, \mathbf{r}\, d^3\mathbf{r}$, which is the natural generalization of the previous expression written in equation (5).

We finish this section by emphasizing the idea that the electric dipole moment, as well as any other higher electric multipole (terms that were not written in the expansion (4)) arises naturally whenever we write approximate expressions for the electric field of localized charge distributions at points very far from the distribution and consider the dominant contributions. Notice that for neutral distributions, the dominant contribution is the electric dipole one.

3. Magnetic dipole moment: a new proposal

Based on our previous discussion, the most natural way of introducing the concept of a magnetic dipole moment of a localized and steady current distribution is to start with the exact expression for the magnetic field $\mathbf{B}(\mathbf{r})$ of the system at a generic point of space P of vector position $\mathbf{r}$, and look for an approximate expression for $\mathbf{B}(\mathbf{r})$ in the case where P is very far from the current distribution. It is really surprising that, as far as the authors' knowledge goes, this procedure is not adopted by any introductory textbook in electromagnetism.

We shall follow here a completely analogous procedure to that used in the previous section. For simplicity, but without any loss of generality, instead of considering steady currents described by a current density vector $\mathbf{j}(\mathbf{r})$, we shall consider steady current loops carrying a given current I. However, we must emphasize that our treatment is quite general in that the current loops to be considered can have arbitrary forms and are not necessarily planar.

The exact expression for the magnetic field created by an arbitrary steady current loop of current I, described by an oriented closed curve C, at an arbitrary point of space (except those belonging to the current loop) is given by the Biot–Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{\mathrm{d}\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\,. \tag{6}$$

For a distant point, or in other words assuming that $r'/r \ll 1$, we may use the same approximation as before (see equation (2)), and write

$$|\mathbf{r} - \mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}' = r\left(1 - \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{r}\right), \tag{7}$$

which leads us to the following approximate result

$$\begin{aligned}
\frac{4\pi}{\mu_0 I}\mathbf{B}(\mathbf{r}) &\simeq \frac{1}{r^3} \oint_C \frac{\mathrm{d}\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{\left(1 - \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r}\right)^3} \\
&\simeq \frac{1}{r^3} \oint_C \mathrm{d}\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')\left(1 + 3\frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{r}\right) \\
&\simeq -\frac{1}{r^3} \oint_C \mathrm{d}\mathbf{r}' \times \mathbf{r}' + \frac{3}{r^3}\left(\oint \mathrm{d}\mathbf{r}'(\hat{\mathbf{r}} \cdot \mathbf{r}')\right) \times \hat{\mathbf{r}},
\end{aligned} \tag{8}$$

where we used the fact that $\left(\oint \mathrm{d}\mathbf{r}'\right) \times \mathbf{r} = \mathbf{0}$, since for a closed curve we have $\oint \mathrm{d}\mathbf{r}' = \mathbf{0}$, and we kept in the integrand only terms up to the linear order in $\mathbf{r}'$.

We now write the last term on the rhs of (8) in a more convenient form. Using the following identity (see the appendix for a simple demonstration)

$$\oint (\mathbf{c}\cdot\mathbf{r}')\,\mathrm{d}\mathbf{r}' = \left(\frac{1}{2}\oint \mathbf{r}'\times\mathrm{d}\mathbf{r}'\right)\times\mathbf{c}\,, \tag{9}$$

with $\mathbf{c}$ being a constant vector, we see that

$$\oint_C \mathrm{d}\mathbf{r}'(\hat{\mathbf{r}}\cdot\mathbf{r}') = \left(\frac{1}{2}\oint_C \mathbf{r}'\times\mathrm{d}\mathbf{r}'\right)\times\hat{\mathbf{r}}\,, \tag{10}$$

and, consequently, we may write

$$\begin{aligned}\frac{3}{r^3}\left(\oint_C \mathrm{d}\mathbf{r}'(\hat{\mathbf{r}}\cdot\mathbf{r}')\right)\times\hat{\mathbf{r}} &= \frac{3}{r^3}\left[\left(\frac{1}{2}\oint_C \mathbf{r}'\times\mathrm{d}\mathbf{r}'\right)\times\hat{\mathbf{r}}\right]\times\hat{\mathbf{r}}\\ &= \frac{3}{r^3}\left[\left(\frac{1}{2}\oint_C \mathbf{r}'\times\mathrm{d}\mathbf{r}'\right)\cdot\hat{\mathbf{r}}\right]\hat{\mathbf{r}} - \frac{3}{r^3}\left(\frac{1}{2}\oint_C \mathbf{r}'\times\mathrm{d}\mathbf{r}'\right),\end{aligned}$$

where we used the identity $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$. Substituting last equation into (8), using that $\mathbf{r}'\times\mathrm{d}\mathbf{r}' = -\mathrm{d}\mathbf{r}'\times\mathbf{r}'$ and defining the vector quantity

$$\mathbf{m} = \frac{I}{2}\oint_C \mathbf{r}'\times\mathrm{d}\mathbf{r}'\,, \tag{11}$$

we finally obtain

$$\mathbf{B}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi}\left[\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3}\right]. \tag{12}$$

Quantity $\mathbf{m}$ is called the magnetic dipole moment of the current distribution, a current loop in the case considered here. Note that in the magnetic case the first term of the expansion is already the dipole contribution and not the monopole one, as in the electric case (since there are no magnetic charges in nature). Note the similarity of (12) with the expression previously obtained for the electric dipole contribution (see equation (4)): the dipole contribution for the magnetic field and the dipole contribution for the electric field can be mapped into each other by the simple exchanges $\epsilon_0 \longleftrightarrow 1/\mu_0$ and $\mathbf{p} \longleftrightarrow \mathbf{m}$.

The usual result for a planar current loop discussed in introductory textbooks may be re-obtained in a straightforward way. For this particular case, the vector product $(1/2)\mathbf{r}\times\mathrm{d}\mathbf{r}$ always has the same direction for all elements of the current loop (except when $\mathbf{r}$ is parallel to $\mathrm{d}\mathbf{r}$, since in this case $\mathbf{r}\times\mathrm{d}\mathbf{r} = \mathbf{0}$) and is equal to $\hat{\mathbf{n}}\,\mathrm{d}A$, where $\mathrm{d}A$ is the infinitesimal area subtended by the element $\mathrm{d}\mathbf{r}$ of the current loop and $\hat{\mathbf{n}}$ is the properly oriented unitary vector normal to the surface containing the current loop. Hence, for a planar current loop we re-obtain for the magnetic dipole moment the well known result $\mathbf{m} = IA\hat{\mathbf{n}}$, where A is the area of the current loop and $\hat{\mathbf{n}}$ is the unit vector normal to the current loop, oriented such that the curve C which describes the current loop is oriented in the direction of the current flow.

4. Torque on an arbitrary current loop

Most textbooks compute the torque exerted on a current loop by an external uniform magnetic field only in very particular cases: rectangular current loops (or circular ones at most) and magnetic fields with particular orientations. This fact motivated us to present here a very general calculation of the torque on a current loop of generic form, not necessarily planar, due to an external uniform magnetic field of arbitrary direction. Despite its general character, our demonstration can be taught in any introductory course in electromagnetism. The torque relative to a point A (base point) acting on a current loop $\mathcal{C}$ due to a uniform magnetic field $\mathbf{B}$ is given by

$$\boldsymbol{\tau}_A = \oint_C (\mathbf{r} - \mathbf{r}_A) \times \mathrm{d}\mathbf{F}\,, \tag{13}$$

where $\mathrm{d}\mathbf{F}$ is the infinitesimal magnetic force acting on an infinitesimal element of the current loop at position $\mathbf{r}$, namely, $\mathrm{d}\mathbf{F} = \mathrm{I}\mathrm{d}\mathbf{r} \times \mathbf{B}$. Before we obtain an expression for this torque, let us show that it is independent of the choice of the base point. With this intention, we write equation (13) with another base point, say, Q, and subtract the equation thus obtained from (13) to obtain

$$\boldsymbol{\tau}_A - \boldsymbol{\tau}_Q = \oint_C (\mathbf{r}_Q - \mathbf{r}_A) \times \mathrm{d}\mathbf{F} = (\mathbf{r}_Q - \mathbf{r}_A) \times \oint_C \mathrm{d}\mathbf{F}\,, \tag{14}$$

where we used that $(\mathbf{r}_Q - \mathbf{r}_A)$ is a constant vector and the distributive property of the vector product. Recalling that $\oint_C \mathrm{d}\mathbf{F} = \oint_C (I\mathrm{d}\mathbf{r} \times \mathbf{B}) = I\left(\oint_C \mathrm{d}\mathbf{r}\right) \times \mathbf{B} = \mathbf{0}$ ($\mathbf{B}$ is uniform), we see that $\boldsymbol{\tau}_A = \boldsymbol{\tau}_Q$. Since A and Q are arbitrary points, we conclude that, in the case under consideration, the torque is independent of the base point. For simplicity, but without any loss of generality, we choose the base point at the origin of our axis system and write the torque on the current loop, relative to the origin, simply as

$$\boldsymbol{\tau} = I \oint_C \mathbf{r} \times (\mathrm{d}\mathbf{r} \times \mathbf{B})\,. \tag{15}$$

We want to compute this torque for a generic current loop, not necessarily planar, under the influence of a uniform magnetic field with an arbitrary direction relative to the current loop. With this goal, we start by using the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, which allows us to cast the previous equation into the form

$$\boldsymbol{\tau} = I \oint_C (\mathbf{r} \cdot \mathbf{B})\mathrm{d}\mathbf{r} - I \oint_C (\mathbf{r} \cdot \mathrm{d}\mathbf{r})\mathbf{B}\,. \tag{16}$$

The last term of the rhs of equation (16) vanishes, since it can be written as a loop integral of a total differential, namely,

$$\oint_C (\mathbf{r} \cdot \mathrm{d}\mathbf{r})\mathbf{B} = \mathbf{B} \oint_C r\,\mathrm{d}r = \mathbf{B} \oint_C \mathrm{d}(r^2/2) = \mathbf{0}\,, \tag{17}$$

so that the torque on the current loop can be written as

$$\boldsymbol{\tau} = I \oint_C (\mathbf{r} \cdot \mathbf{B})\,\mathrm{d}\mathbf{r}\,. \tag{18}$$

Equation (18) can be written in an equivalent but convenient form. Denoting by $\mathbf{e}_i$, $i = 1, 2, 3$, the unitary vectors of the Cartesian basis, it takes the form

$$\begin{aligned}
\boldsymbol{\tau} &= I \sum_{i,j} B_i \hat{\mathbf{e}}_j \oint_C x_i\,\mathrm{d}x_j \\
&= I \sum_{i,j} B_i \hat{\mathbf{e}}_j \left[\oint_C \mathrm{d}(x_i x_j) - \oint_C x_j\,\mathrm{d}x_i \right] \\
&= -I \sum_{i,j} \oint (B_i\,\mathrm{d}x_i)\, x_j \hat{\mathbf{e}}_j \\
&= -I \oint_C (\mathbf{B} \cdot \mathrm{d}\mathbf{r})\,\mathbf{r}\,,
\end{aligned} \tag{19}$$

where from passing from the first line to the second we made an integration by parts, and from the second line to the third, we just used the fact that a closed line integral of a total derivative vanishes. Multiplying both equations (18) and (19) by 1/2 and summing the results thus obtained, we get

$$\boldsymbol{\tau} = \frac{I}{2}\left\{ \oint_C [(\mathbf{r}\cdot\mathbf{B})\,\mathrm{d}\mathbf{r} - (\mathbf{B}\cdot\mathrm{d}\mathbf{r})\,\mathbf{r}] \right\}. \tag{20}$$

From the identity $(\mathbf{r}\times\mathrm{d}\mathbf{r})\times\mathbf{B} = (\mathbf{r}\cdot\mathbf{B})\mathrm{d}\mathbf{r} - (\mathbf{B}\cdot\mathrm{d}\mathbf{r})\mathbf{r}$, equation (20) can be written as

$$\boldsymbol{\tau} = \frac{I}{2}\oint_C (\mathbf{r}\times\mathrm{d}\mathbf{r})\times\mathbf{B}. \tag{21}$$

Since $\mathbf{B}$ is a uniform field, we may take $\mathbf{B}$ outside the integral to cast the previous equation into the form

$$\boldsymbol{\tau} = \left(\frac{I}{2}\oint_C \mathbf{r}\times\mathrm{d}\mathbf{r}\right)\times\mathbf{B} = \mathbf{m}\times\mathbf{B}, \tag{22}$$

where we identified $\frac{I}{2}\oint_C \mathbf{r}\times\mathrm{d}\mathbf{r}$ as the magnetic dipole moment of the current loop C, introduced in the previous section. It is worth noting that in the previous demonstration that the closed curve C describing the current loop is a planar curve was not required. Further, the direction of the external magnetic field was not specified. In this sense, our result generalizes those found in the usual textbooks.

5. Conclusion and final remarks

In this work we made a proposal of how to introduce the concept of the magnetic dipole moment in introductory courses on electromagnetism following a procedure analogous to that used for introducing the electric dipole moment. Essentially, the idea is to start with a localized and steady current distribution and look for an approximate expression for the corresponding magnetic field at points far from the current distribution. This was done directly from the Biot–Savart law. In this way, the students can understand the similarities and differences between the two cases and are prepared in a natural way to understand the underlying ideas of the so-called multipole expansion to be studied in more advanced courses. We also presented a simple but general demonstration for the torque exerted by a uniform magnetic field on an arbitrary current loop, not necessarily planar. Our result reduces to the usual one for planar current loops of particular forms and orientations found in introductory textbooks. We think the results presented here are accessible for beginners who already have a basic knowledge of vector calculus and we hope that both our results will be useful in introductory courses on electromagnetism. We should mention here that in most advanced textbooks, see for instance Feynman's [16] or Griffiths' textbook [17], the magnetic dipole moment is also introduced with the aid of a Taylor expansion as in this paper, but these textbooks do that by working with the vector potential $\mathbf{A}$ which is not discussed in introductory courses. Also, regardless of their convenience, electromagnetic potentials are auxiliary quantities so that it is interesting to obtain physical results working directly with the electromagnetic fields. Finally, we should emphasize that the results obtained in section 3 can be generalized straightforwardly for any steady distribution described by a vector current density $\mathbf{j}$. We suggest that the interested reader make such a generalization and arrive, after following a procedure analogous to that of section 3, at the result $\mathbf{m} = \frac{1}{2}\int_{\mathcal{R}}(\mathbf{r}\times\mathbf{j}(\mathbf{r}))\,\mathrm{d}^3\mathbf{r}$, which generalizes (11) for the magnetic dipole moment of a generic steady current distribution.

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Appendix. Demonstration of equation (9)

In this appendix we demonstrate the vector identity appearing in (9). We start by writing the Stokes theorem, $\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, \mathrm{dA}$, where S is an arbitrary surface and ∂S its border. For a particular vector field $\mathbf{F} = \mathbf{c}f$, where $\mathbf{c}$ is an arbitrary constant vector and f, an arbitrary scalar function, the previous equation takes the form

$$\mathbf{c} \cdot \oint_{\partial S} f\, \mathrm{d}\mathbf{r} = -\int_S (\mathbf{c} \times \nabla f) \cdot \hat{\mathbf{n}}\, \mathrm{d}A\,, \tag{A.1}$$

where we used the identity $\nabla \times (\mathbf{c}f) = f\nabla \times \mathbf{c} - \mathbf{c} \times \nabla f$ and $\nabla \times \mathbf{c} = \mathbf{0}$. Recalling that $(\mathbf{c} \times \nabla f) \cdot \hat{\mathbf{n}} = \mathbf{c} \cdot (\nabla f \times \hat{\mathbf{n}})$, (A.1) can be written as $\mathbf{c} \cdot \oint_{\partial S} f\, \mathrm{d}\mathbf{r} = -\mathbf{c} \cdot \int_S \nabla f \times \hat{\mathbf{n}}\, \mathrm{d}A$. Since $\mathbf{c}$ is an arbitrary vector, we obtain the following identity

$$\oint_{\partial S} f\, \mathrm{d}\mathbf{r} = -\int_S \nabla f \times \hat{\mathbf{n}}\, \mathrm{d}A\,. \tag{A.2}$$

Taking $f = \mathbf{b} \cdot \mathbf{r}$, with $\mathbf{b}$ a constant vector, and using that $\nabla f = \nabla(\mathbf{b} \cdot \mathbf{r}) = \mathbf{b}$, we have

$$\oint_{\partial S} (\mathbf{b} \cdot \mathbf{r})\, \mathrm{d}\mathbf{r} = -\mathbf{b} \times \left(\oint_S \hat{\mathbf{n}}\, \mathrm{d}A \right). \tag{A.3}$$

Using that $\hat{\mathbf{n}}\, \mathrm{dA} = \frac{1}{2}\mathbf{r} \times \mathrm{d}\mathbf{r}$ and the antisymmetry of the vector product, we finally obtain

$$\oint_{\partial S} (\mathbf{b} \cdot \mathbf{r})\, \mathrm{d}\mathbf{r} = \left(\frac{1}{2} \oint_{\partial S} \mathbf{r} \times \mathrm{d}\mathbf{r} \right) \times \mathbf{b}\,, \tag{A.4}$$

which is precisely the vector identity written in equation (9).

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