On the exact electric and magnetic fields of an electric dipole

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We make a multipole expansion directly in Jefimenko's equations to obtain the exact expressions for the electric and magnetic fields of an electric dipole with an arbitrary time dependence. Some comments are made about the usual derivations in most undergraduate and graduate textbooks and in literature.
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The problem of finding an analytic expression for the electric field of a localized but arbitrary static charge distribution is quite involved. Due to the difficulty in obtaining exact solutions, numerical methods and approximate theoretical methods have been developed. One of the most important examples of the latter is the multipole expansion method. For the origin inside the distribution, the multipole expansion method gives the field outside the distribution as a superposition of fields, each of which can be interpreted as the electrostatic field of a multipole located at the origin (see, for instance, Ref. 1). The first three terms of the multipole expansion correspond to the fields of a monopole, a dipole, and a quadrupole, respectively. For an arbitrary distribution with a vanishing total charge, the first term in the multipole expansion is the field of the electric dipole of the distribution. This property is an example of the important role played by the field of an electric dipole.

For (localized) charge and current distributions with arbitrary time dependence, the multipole expansion method is a powerful technique for calculating the electromagnetic fields of the system. In the study of radiation fields of simple systems, such as antennas, it is common to make a multipole expansion and to keep only the first few terms. It is worth emphasizing that in contrast to what happens in the case of static charge distributions, the leading contribution to the radiation fields comes from the electric dipole term since there is no monopole radiation due to charge conservation.

The electric field of an electric dipole has three contributions—the static zone contribution proportional to $1/r^3$, the intermediate zone contribution proportional to $1/r^2$, and the far zone or radiation contribution proportional to $1/r$, where r is the distance from the origin. (As we shall see, the corresponding magnetic field has only two contributions.) A general derivation of the exact electric and magnetic fields of an electric dipole is lacking from most textbooks.[2–12] In some cases, the textbooks obtain the exact electromagnetic fields for the electric dipole by assuming a harmonic time dependence for the sources[2–5] or by assuming a particular model for the charge and current distributions.[6–8] In other cases, the textbooks are interested only in the radiation fields.[9–12]

A general derivation of the exact electric and magnetic fields of an electric dipole with arbitrary time dependence can be found in Ref. 13. Heras first presented a new version of Jefimenko's equations in a material medium[14] with polarization $\mathbf{P}$ and magnetization $\mathbf{M}$ and then obtained the exact dipole fields.

The purpose of this note is to provide a direct derivation of the exact electric and magnetic fields of an electric dipole with arbitrary time dependence without making any assumptions for the (localized) charge and current distributions of the system. Our starting point will be Jefimenko's equations and our procedure consists of making a multipole expansion up to the desired order. Our procedure does not require Jefimenko's equation in matter, which makes our derivation a very simple one.

Jefimenko's equations for arbitrary but localized sources are given by[15–19]

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\left\{\int d\mathbf{r}'\frac{[\rho(\mathbf{r}',t')]\mathbf{R}}{R^3} + \int d\mathbf{r}'\frac{[\dot{\rho}(\mathbf{r}',t')]\mathbf{R}}{cR^2} - \int d\mathbf{r}'\frac{[\dot{\mathbf{J}}(\mathbf{r}',t')]}{c^2R}\right\}, \tag{1}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\left\{\int d\mathbf{r}'\frac{[\mathbf{J}(\mathbf{r}',t')]\times\mathbf{R}}{R^3} + \int d\mathbf{r}'\frac{[\dot{\mathbf{J}}(\mathbf{r}',t')]\times\mathbf{R}}{cR^2}\right\}, \tag{2}$$

where the integrals are over all space, $R=|\mathbf{R}|=|\mathbf{r}-\mathbf{r}'|$, the overdot means time derivative [for example, $\dot{\rho}(\mathbf{r}',t') = \partial\rho(\mathbf{r}',t')/\partial t'$], and the brackets $[\ldots]$ mean that the quantities inside them must be evaluated at the retarded time $t' = t-R/c$.

We make a multipole expansion, which consists of making an expansion in powers of r'/r, which is valid outside the charge and current distributions. Our final result will be valid only outside the sources. Because we are interested in the dipole fields, we shall retain only terms up to linear order in $\mathbf{r}'$. Therefore, we use the following Taylor expansions:

$$\frac{1}{R} \simeq \frac{1}{r} + \frac{(\hat{\mathbf{r}}\cdot\mathbf{r}')}{r^2}, \tag{3}$$

$$\frac{\mathbf{R}}{R^2} \simeq \frac{\hat{\mathbf{r}}}{r} + \frac{2(\hat{\mathbf{r}}\cdot\mathbf{r}')\hat{\mathbf{r}}}{r^2} - \frac{\mathbf{r}'}{r^2}, \tag{4}$$

$$\frac{\mathbf{R}}{R^3} \simeq \frac{\hat{\mathbf{r}}}{r^2} + \frac{3(\hat{\mathbf{r}}\cdot\mathbf{r}')\hat{\mathbf{r}}}{r^3} - \frac{\mathbf{r}'}{r^3}, \tag{5}$$

$$t-\frac{R}{c}\simeq t_0+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}, \tag{6}$$

where $t_0=t-r/c$ is the retarded time with respect to the origin. Analogously, we expand the source terms about $t'=t_0$ and keep terms only up to linear order in $\mathbf{r}'$. The results are

$$\begin{aligned}[\rho(\mathbf{r}',t')]&=\rho(\mathbf{r}',t-R/c)\\&\simeq\rho(\mathbf{r}',t_0+\hat{\mathbf{r}}\cdot\mathbf{r}'/c)\simeq\rho(\mathbf{r}',t_0)+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\dot{\rho}(\mathbf{r}',t_0),\end{aligned} \tag{7}$$

$$\begin{aligned}[\mathbf{J}(\mathbf{r}',t')]&=\mathbf{J}(\mathbf{r}',t-R/c)\\&\simeq\mathbf{J}(\mathbf{r}',t_0+\hat{\mathbf{r}}\cdot\mathbf{r}'/c)\simeq\mathbf{J}(\mathbf{r}',t_0)+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\dot{\mathbf{J}}(\mathbf{r}',t_0),\end{aligned} \tag{8}$$

where we used Eq. (6). We can make analogous expansions to obtain approximate expressions for the time derivatives of the sources, namely,

$$[\dot{\rho}(\mathbf{r}',t')]\simeq\dot{\rho}(\mathbf{r}',t_0)+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\ddot{\rho}(\mathbf{r}',t_0), \tag{9}$$

$$[\dot{\mathbf{J}}(\mathbf{r}',t')]\simeq\dot{\mathbf{J}}(\mathbf{r}',t_0)+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\ddot{\mathbf{J}}(\mathbf{r}',t_0). \tag{10}$$

For convenience, we denote the three integrals that appear on the right-hand side of Eq. (1) by $I_1^{(E)}$, $I_2^{(E)}$, and $I_3^{(E)}$. Similarly, we denote by $I_1^{(B)}$ and $I_2^{(B)}$ the two integrals that appear on the right-hand side of Eq. (2).

We must calculate these integrals up to the desired order. We first consider $I_1^{(E)}$, substitute Eqs. (5) and (7) into the expression for $I_1^{(E)}$, and obtain

$$\begin{aligned}I_1^{(E)}&\simeq\int d\mathbf{r}'\left\{\rho(\mathbf{r}',t_0)+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\dot{\rho}(\mathbf{r}',t_0)\right\}\\&\quad\times\left\{\frac{\hat{\mathbf{r}}}{r^2}+\frac{3(\hat{\mathbf{r}}\cdot\mathbf{r}')\hat{\mathbf{r}}}{r^3}-\frac{\mathbf{r}'}{r^3}\right\}\end{aligned} \tag{11}$$

$$\begin{aligned}&\simeq\frac{\hat{\mathbf{r}}}{r^2}\left\{\int d\mathbf{r}'\rho(\mathbf{r}',t_0)\right\}+\frac{3\hat{\mathbf{r}}}{r^3}\left\{\hat{\mathbf{r}}\cdot\int d\mathbf{r}'\rho(\mathbf{r}',t_0)\mathbf{r}'\right\}\\&\quad-\frac{1}{r^3}\left\{\int d\mathbf{r}'\rho(\mathbf{r}',t_0)\mathbf{r}'\right\}+\frac{\hat{\mathbf{r}}}{cr^2}\left\{\hat{\mathbf{r}}\cdot\int d\mathbf{r}'\dot{\rho}(\mathbf{r}',t_0)\mathbf{r}'\right\},\end{aligned} \tag{12}$$

where quadratic terms in $\mathbf{r}'$ are neglected. The integral in the first term on the right-hand side of Eq. (12) is the total charge of the distribution, which is independent of time due to charge conservation. This term corresponds to the monopole term of the multipole expansion because it is the field created by a point charge $Q\equiv\int\rho(\mathbf{r}',t_0)d\mathbf{r}'$ fixed at the origin. This term does not contribute to the dipole field. We can identify the electric dipole moment of the distribution at time t_0 in the next two integrals on the right-hand side of Eq. (12), namely, $\mathbf{p}(t_0)=\int d\mathbf{r}'\rho(\mathbf{r}',t_0)\mathbf{r}'$. After we interchange the time derivative with the integration in the last integral on the right-hand side of Eq. (12), we can identify the time derivative of the electric dipole moment of the distribution at time t_0, namely, $\dot{\mathbf{p}}(t_0)=(d/dt)\int d\mathbf{r}'\rho(\mathbf{r}',t_0)\mathbf{r}'$. Hence, $I_1^{(E)}$ is given by

$$I_1^{(E)}\simeq\frac{Q\hat{\mathbf{r}}}{r^2}+\frac{3[\hat{\mathbf{r}}\cdot\mathbf{p}(t_0)]\hat{\mathbf{r}}-\mathbf{p}(t_0)}{r^3}+\frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)}{cr^2}. \tag{13}$$

We next use an analogous procedure and substitute into the expression for $I_2^{(E)}$ the approximations in Eqs. (4) and (9). The result is

$$I_2^{(E)}\simeq\frac{2[\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)]\hat{\mathbf{r}}-\dot{\mathbf{p}}(t_0)}{cr^2}+\frac{[\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0)]\hat{\mathbf{r}}}{c^2r}. \tag{14}$$

To calculate $I_3^{(E)}$, $I_1^{(B)}$, and $I_2^{(B)}$, it is convenient to write the Cartesian basis vectors as $\hat{\mathbf{e}}_i=\boldsymbol{\nabla}'x_i'$ $(i=1,2,3)$ so that any vector $\mathbf{b}$ can be written in the form $\mathbf{b}=b_i\hat{\mathbf{e}}_i=(\mathbf{b}\cdot\hat{\mathbf{e}}_i)\hat{\mathbf{e}}_i=(\mathbf{b}\cdot\boldsymbol{\nabla}'x_i')\hat{\mathbf{e}}_i$, where the Einstein convention of summation over repeated indices is assumed. As it will become clear, it suffices to make the approximations $R\to r$ and $[\dot{\mathbf{J}}(\mathbf{r}',t')]\to\dot{\mathbf{J}}(\mathbf{r}',t_0)$ in the integrand. Hence, we have

$$\begin{aligned}I_3^{(E)}&\simeq-\frac{1}{c^2r}\int d\mathbf{r}'\dot{\mathbf{J}}(\mathbf{r}',t_0)\\&=-\frac{1}{c^2r}\hat{\mathbf{e}}_i\int d\mathbf{r}'\{\dot{\mathbf{J}}(\mathbf{r}',t_0)\cdot\boldsymbol{\nabla}'x_i'\}\end{aligned} \tag{15a}$$

$$\begin{aligned}&=\frac{1}{c^2r}\hat{\mathbf{e}}_i\int d\mathbf{r}'x_i'\{\boldsymbol{\nabla}'\cdot\dot{\mathbf{J}}(\mathbf{r}',t_0)\}\\&=-\frac{1}{c^2r}\hat{\mathbf{e}}_i\int d\mathbf{r}'x_i'\ddot{\rho}(\mathbf{r}',t_0),\end{aligned} \tag{15b}$$

where in passing from Eq. (15a) to Eq. (15b) we integrated by parts and discarded the surface term because the integration is over all space and the sources are localized. In the last step we used the continuity equation. The second time derivative of the electric dipole moment of the distribution at time t_0 appears in Eq. (14), and thus we obtain

$$I_3^{(E)}\simeq-\frac{\ddot{\mathbf{p}}(t_0)}{c^2r}. \tag{16}$$

We now see why it was not necessary to go beyond zeroth order in the expansion of $\dot{\mathbf{J}}(\mathbf{r}',t')$ in the calculation of $I_3^{(E)}$. The electric dipole moment of the distribution is already first order in $\mathbf{r}'$ so that the next order term would contribute to the next terms of the multipole expansion, namely, the magnetic dipole term and the electric quadrupole term.

The integrals $I_1^{(B)}$ and $I_2^{(B)}$ will give the contribution to the magnetic field of the electric dipole term. We proceed as we did for $I_3^{(E)}$, namely, it suffices to use only the zeroth order approximation for the expansions of the current and its time derivative. We have

$$I_1^{(B)} \simeq -\frac{\hat{\mathbf{r}}}{r^2} \times \left\{ \int d\mathbf{r}' \mathbf{J}(\mathbf{r}', t_0) \right\} \tag{17a}$$

$$= -\frac{\hat{\mathbf{r}}}{r^2} \times \hat{\mathbf{e}}_i \int d\mathbf{r}' \{ \mathbf{J}(\mathbf{r}', t_0) \cdot \boldsymbol{\nabla}' x_i' \} \tag{17b}$$

$$= \frac{\hat{\mathbf{r}}}{r^2} \times \hat{\mathbf{e}}_i \int d\mathbf{r}' x_i' \{ \boldsymbol{\nabla}' \cdot \mathbf{J}(\mathbf{r}', t_0) \} \tag{17c}$$

$$= -\frac{\hat{\mathbf{r}}}{r^2} \times \hat{\mathbf{e}}_i \int d\mathbf{r}' x_i' \dot{\rho}(\mathbf{r}', t_0) = \frac{\dot{\mathbf{p}}(t_0) \times \hat{\mathbf{r}}}{r^2}, \tag{17d}$$

where in the last step we identified the time derivative of the electric dipole moment of the distribution. As before, it is straightforward to show that

$$I_2^{(B)} \simeq \frac{\ddot{\mathbf{p}}(t_0) \times \hat{\mathbf{r}}}{cr}. \tag{18}$$

We substitute the results in Eqs. (13), (14), and (16) into Eq. (1), discard the monopole term, substitute Eqs. (17d) and (18) into Eq. (2), and obtain the exact electric and magnetic fields associated with the electric dipole term of an arbitrary (localized) distribution of charges and currents,

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{3[\hat{\mathbf{r}} \cdot \mathbf{p}(t_0)]\hat{\mathbf{r}} - \mathbf{p}(t_0)}{r^3} + \frac{3[\hat{\mathbf{r}} \cdot \dot{\mathbf{p}}(t_0)]\hat{\mathbf{r}} - \dot{\mathbf{p}}(t_0)}{cr^2} + \frac{\hat{\mathbf{r}} \times [\hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t_0)]}{c^2 r} \right\}, \tag{19}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi} \left\{ \frac{\dot{\mathbf{p}}(t_0) \times \hat{\mathbf{r}}}{r^2} + \frac{\ddot{\mathbf{p}}(t_0) \times \hat{\mathbf{r}}}{cr} \right\}, \tag{20}$$

which are the desired fields. Equation (19) is given, but not derived, in Ref. 20.

The last terms on the right-hand side of Eqs. (19) and (20) are proportional to $1/r$ and correspond to the radiation fields of an electric dipole. For the idealized case of a point electric dipole fixed at the origin, Eqs. (19) and (20) are exact for every point in space, except the origin, where the fields are singular. Because we are interested in the fields outside the dipole, we shall not be concerned with these singular terms. (A discussion of these terms for a static point electric dipole can be found in Ref. 2, Chap. 3.)

As an important special case, we consider an electric dipole with harmonic time dependence. We substitute $\mathbf{p}(t) = \mathbf{p}_0 e^{-i\omega t}$ into Eqs. (19) and (20) and obtain the well known results,[2]

$$\mathbf{E}(\mathbf{r},t) = \frac{e^{-i\omega t}}{4\pi\epsilon_0} \left\{ k^2 (\hat{\mathbf{r}} \times \mathbf{p}_0) \times \hat{\mathbf{r}} \frac{e^{ikr}}{r} + [3(\hat{\mathbf{r}} \cdot \mathbf{p}_0)\hat{\mathbf{r}} - \mathbf{p}_0] \left(\frac{1}{r^3} - \frac{ik}{r^2} \right) e^{ikr} \right\}, \tag{21}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0 e^{-i\omega t}}{4\pi} \left\{ ck^2 (\hat{\mathbf{r}} \times \mathbf{p}_0) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr} \right) \right\}, \tag{22}$$

where $k = \omega / c$. In Eq. (19) note the presence of a term proportional to r^{-3} and $\mathbf{p}(t_0)$, which is a characteristic of the field of a static electric dipole, except that here the electric dipole moment is evaluated at the retarded time t_0. This term dominates in the near zone, where $d \ll r \ll \lambda$, where d is a typical length scale of the source and λ is the wavelength of the electromagnetic field. Note the absence of such a term in Eq. (20), which is expected because a static electric dipole does not create a magnetic field. In the radiation zone, where $d \ll \lambda \ll r$, the dominant terms are proportional to r^{-1} and to the second time derivative of the electric dipole moment. The transverse character of the radiation fields is evident.

The results in Eqs. (19) and (20) could have been obtained if, instead of Jefimenko's Eq. (1) for the electric field, we had used the equivalent Panofsky–Phillips equation for the electric field[11] given by

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0} \left\{ \int d\mathbf{r}' \frac{[\rho(\mathbf{r}',t')]\hat{\mathbf{R}}}{R^2} + \int d\mathbf{r}' \frac{([\mathbf{J}(\mathbf{r}',t')] \cdot \hat{\mathbf{R}})\hat{\mathbf{R}} + ([\mathbf{J}(\mathbf{r}',t')] \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}}}{cR^2} \right\} + \frac{1}{4\pi\epsilon_0} \left\{ \int d\mathbf{r}' \frac{([\dot{\mathbf{J}}(\mathbf{r}',t')] \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}}}{c^2 R} \right\}. \tag{23}$$

Equation (23) is equivalent to Eq. (1), as it was shown explicitly in Ref. 21, but note that Eq. (23) is more convenient as a starting point for calculating multipole radiation fields.[22] All multipole radiation fields can be obtained from the last terms of the right-hand side of Eqs. (2) and (23). The dipole radiation fields in Eqs. (19) and (20) are just the first order contributions of the multipole expansion for the radiation fields. The next order contributions, namely, the radiation fields of a magnetic dipole and an electric quadrupole, may be similarly obtained without much more effort.[22]

We leave for the interested reader the problem of obtaining the exact electric field of an electric dipole with arbitrary time dependence from Eq. (23) instead of Eq. (1). The exact electric and magnetic fields of higher multipoles such as the magnetic dipole and electric quadrupole can also be obtained following our approach. In this case, higher order terms in $\mathbf{r}'$ must be kept.

We have calculated the exact electric and magnetic fields of an electric dipole with arbitrary time dependence, in contrast to the usual calculations where a harmonic time dependence is assumed. Most textbooks use the electromagnetic potentials. We have instead implemented a multipole expansion directly from Jefimenko's equations. Besides the simplicity of our method, it increases the number of problems that can be attacked directly by using Jefimenko's equations (see also Refs. 23–28).

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