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LETTERS AND COMMENTS

Another example of classical renormalization in electrostatics

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Abstract
A simple example of renormalization in electrostatics accessible to undergraduate students is presented as a complement to a letter recently published in this journal.

In a recent letter to this journal, Corbò [1] called the reader's attention to the fact that renormalization techniques can be applied to classical fields. As examples, two problems involving the renormalization of the electrostatic potential are discussed. The following example should be considered as a complement to the ones presented in [1] and discusses the renormalization of electrostatic energy.

Suppose that from a classical point of view an atom is modelled in the following way: a central pointlike electric charge whose magnitude is equal to Ze, where Z is the atomic number and $e = 1.6 \times 10^{-19}$ C is the elementary *quantum* of electric charge, surrounded by a concentric thin spherical shell of radius R and electric charge equal to $-Ze$. The partial or total ionization of this classical atom is equivalent to the removal of part of or the entire negative charge from the thin spherical shell. Mathematically, we can express this process by considering the substitution $-Ze \to -Ze(1-\alpha)$, where α is a real number in the closed interval [0, 1]. Our goal will be to show that variation of the electrostatic energy is given by the (renormalized) expression

$$\Delta U = U_{\text{final}} - U_{\text{initial}} = \frac{1}{2}\frac{\alpha^2 Z^2 e^2}{4\pi\epsilon_0 R}. \tag{1}$$

In the region $0 < r < R$, the electric field is that of a point charge

$$\mathbf{E} = \frac{Ze}{4\pi\epsilon_0 r^2}\hat{\mathbf{e}}_r, \qquad 0 < r < R. \tag{2}$$

The contribution of the spherical shell to the inner electric field is zero as we can easily prove by making use of Gauss' law and the spherical symmetry of the problem. In the region $r > R$, using again Gauss' law and the spherical symmetry argument, we can prove that the electric field is zero. The electrostatic energy can be calculated with the formula

$$U = \frac{\epsilon_0}{2} \int_{\mathcal{R}} \mathbf{E}^2 \, \mathrm{d}v. \tag{3}$$

where $\mathcal{R}$ is the region of integration, $\mathrm{d}v$ is the element of volume and ϵ_0 is the vacuum permittivity constant. The electrostatic energy before ionization is

$$\begin{aligned} U &= \frac{\epsilon_0}{2} \int_0^R \left(\frac{Ze}{4\pi \epsilon_0 r^2} \right)^2 4\pi r^2 \, \mathrm{d}r = \frac{1}{2} \frac{Z^2 e^2}{4\pi \epsilon_0} \int_0^R \frac{\mathrm{d}r}{r^2} \\ &= \frac{1}{2} \frac{Z^2 e^2}{4\pi \epsilon_0} \left[-\frac{1}{r} \right]_0^R . \end{aligned} \tag{4}$$

As expected, we have a singular point at $r = 0$ because the function $1/r$ diverges at the origin. In order to circumvent this problem, we introduce a finite non-null radius δ (i.e. a regularization parameter) for the point charge and write

$$U = \frac{1}{2} \frac{Z^2 e^2}{4\pi \epsilon_0} \int_\delta^R \frac{\mathrm{d}r}{r^2} = \frac{1}{2} \frac{Z^2 e^2}{4\pi \epsilon_0} \left[-\frac{1}{r} \right]_\delta^R . \tag{5}$$

It follows that

$$U_{\text{initial}} = -\frac{1}{2} \frac{Z^2 e^2}{4\pi \epsilon_0 R} + \frac{1}{2} \frac{Z^2 e^2}{4\pi \epsilon_0 \delta}. \tag{6}$$

We should not worry about the δ-dependent term. At the end of the calculation this term will be naturally cancelled out. When the 'atom' is ionized, part of the charge of the shell is removed to infinity—this is our definition of ionization—which means that the electric field in the region $r > R$ is no longer zero! The net charge involved by a spherical Gaussian surface with radius $r > R$ will be $q(r) = Ze - Ze(1 - \alpha) = \alpha Ze$. Using Gauss' law once more, we obtain

$$\mathbf{E}(r) = \frac{\alpha Ze}{4\pi \epsilon_0 r^2} \hat{\mathbf{e}}_r, \qquad R < r < \infty. \tag{7}$$

The electric field for $0 < r < R$ continues to be that of a point charge. Therefore, the final electrostatic energy will be

$$U_{\text{final}} = -\frac{1}{2} \frac{Z^2 e^2}{4\pi \epsilon_0 R} + \frac{1}{2} \frac{Z^2 e^2}{4\pi \epsilon_0 \delta} + \frac{\epsilon_0}{2} \int_R^\infty \left(\frac{\alpha Ze}{4\pi \epsilon_0 r^2} \right)^2 4\pi r^2 \, \mathrm{d}r. \tag{8}$$

The variation of the electrostatic energy ΔU after evaluating the integral is

$$\Delta U = U_{\text{final}} - U_{\text{initial}} = \frac{1}{2} \frac{\alpha^2 Z^2 e^2}{4\pi \epsilon_0 R}, \tag{9}$$

which is the finite result we were looking for. Observe that the additional constant introduced to control the divergence at the origin does not influence the final result. As is well known, this divergence is associated with the concept of point charge. Simple examples in a conceptually simpler context, as the ones discussed in [1] and the one discussed here, can be the best way to introduce a difficult concept to our students.

Reference

[1] Corbò G 2010 Renormalization in classical field theory *Eur. J. Phys.* **31** L5–8