Multipole radiation fields from the Jefimenko equation for the magnetic field and the Panofsky-Phillips equation for the electric field

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We show how to obtain the first multipole contributions to the electromagnetic radiation emitted by an arbitrary localized source directly from the Jefimenko equation for the magnetic field and the Panofsky-Phillips equation for the electric field. This procedure avoids the unnecessary calculation of the electromagnetic potentials.
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I. INTRODUCTION

The derivation found in most textbooks of the electromagnetic fields generated by arbitrary sources in vacuum starts by calculating the corresponding electromagnetic potentials (see, for instance, Ref. 1 and 2, or 3). After the retarded potentials are obtained (assuming the Lorenz gauge) the electromagnetic fields are calculated with the aid of the relations $\mathbf{E}=-\nabla\Phi-(1/c)(\partial\mathbf{A}/\partial t)$ and $\mathbf{B}=\nabla\times\mathbf{A}$ (we use Gaussian units). The resulting expressions for the fields are usually called Jefimenko's equations because they appeared for the first time in the textbook by Jefimenko.[4] Jefimenko's equations are obtained in Ref. 1 from the retarded potentials and are obtained directly from Maxwell's equations in Ref. 5. (Heras[6] had already derived Jefimenko's equations directly from Maxwell's equations.) References 8 and 7 obtain a less common form of Jefimenko's equations for the electric field, but this form is more convenient for studying radiation.

Griffiths and Heald[9] illustrate Jefimenko's equations by obtaining the standard Liénard-Wiechert fields for a point charge. Ton[10] provides an alternative derivation of Jefimenko's equations and three applications, including that of a point charge in arbitrary motion. Heras has generalized Jefimenko's equations to include magnetic monopoles and obtained the electric and magnetic fields of a particle with both electric and magnetic charge in arbitrary motion.[11] He has also discussed Jefimenko's equations in material media to obtain the electric and magnetic fields of a dipole in arbitrary motion[12] and has derived Jefimenko's equations from Maxwell's equations using the retarded Green function of the wave equation.[6]

The main purpose of this paper is to enlarge the list of problems that are solved directly from Jefimenko's equations (or the equivalent). Our procedure avoids completely the use of electromagnetic potentials. Specifically, we shall obtain the electric dipole, the magnetic dipole, and the electric quadrupole terms of the multipole expansion due to the radiation fields of an arbitrary localized source.

II. JEFIMENKO'S EQUATIONS FROM MAXWELL'S EQUATIONS

In this section we present three methods of calculating Jefimenko's equations directly from Maxwell's equations. The first method closely follows Ref. 5. The second method makes use of a Fourier transformation as discussed by Ref. 16. For our purposes it suffices to do a Fourier transformation only in the temporal coordinate. We shall see that this method, although longer than the previous one, avoids any possibility of misleading manipulations with retarded quantities. The subtleties in calculations involving retarded quantities have been discussed in Refs. 13–15. We then present an alternative method that is a variation of the first one; the main difference is the order of integration.

A. Direct calculation using the retarded Green function

The following approach is similar to that in Refs. 5, 6, and 13. Maxwell's equations with sources in vacuum are given by

$$\nabla\cdot\mathbf{E}=4\pi\rho, \tag{1}$$

$$\nabla\cdot\mathbf{B}=0, \tag{2}$$

$$\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \tag{3}$$

$$\nabla\times\mathbf{B}=\frac{4\pi}{c}\mathbf{J}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}. \tag{4}$$

If we take the curl of Eq. (3) and the time derivative of Eq. (4), we obtain

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}=4\pi\left(\nabla\rho+\frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t}\right), \tag{5}$$

where we have used $\nabla\times(\nabla\times\mathbf{F})=\nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F}$, with $\nabla^2\mathbf{F}=(\nabla^2F_x)\hat{\mathbf{x}}+(\nabla^2F_y)\hat{\mathbf{y}}+(\nabla^2F_z)\hat{\mathbf{z}}$ with $\mathbf{F}$ an arbitrary function. Similarly, we obtain for the magnetic field

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B}=-\frac{4\pi}{c}\mathbf{J}. \tag{6}$$

The solutions of Eqs. (5) and (6) can be obtained with the aid of the retarded Green function $G_{\mathrm{ret}}(\mathbf{x},t;\mathbf{x}',t')$, which satisfies the inhomogeneous differential equation

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G_{\mathrm{ret}}(\mathbf{x},t;\mathbf{x}',t')=\delta(\mathbf{x}-\mathbf{x}')\delta(t-t'), \tag{7}$$

and is zero for $t-t'<0$. The solution for G_{ret} for $t-t'>0$ is given by[3]

$$G_{\mathrm{ret}}(\mathbf{x},t;\mathbf{x}',t') = -\frac{1}{4\pi}\frac{\delta(t'-(t-|\mathbf{x}-\mathbf{x}'|/c))}{|\mathbf{x}-\mathbf{x}'|}$$
$$= -\frac{1}{4\pi}\frac{\delta(t'-(t-R/c))}{R}, \tag{8}$$

where $R \equiv |\mathbf{R}| = |\mathbf{x}-\mathbf{x}'|$. With the help of Eq. (8) the solution of Eq. (5) can be written as

$$\mathbf{E}(\mathbf{x},t) = -\int \frac{d\mathbf{x}'}{R}\int dt'\,\delta(t'-(t-R/c)) \times\left(\nabla'\rho(\mathbf{x}',t') + \frac{1}{c^2}\frac{\partial \mathbf{J}(\mathbf{x}',t')}{\partial t'}\right) \tag{9}$$

$$= -\int d\mathbf{x}'\frac{[\nabla'\rho]}{R} - \int d\mathbf{x}'\frac{[\dot{\mathbf{J}}]}{c^2R}, \tag{10}$$

where the notation $[\cdots]$ means that the quantity inside the brackets is a function of $\mathbf{x}'$ and is evaluated at the retarded time $t' = t - |\mathbf{x}-\mathbf{x}'|/c$.

At this point much care must be taken, because $\nabla'[\rho] \neq [\nabla'\rho]$. Due to a bad choice of notation, the author in Ref. 13 incorrectly used the quantity $[\nabla'\rho]$ as if it were $\nabla'[\rho]$ and, after an integration by parts, an incomplete result for the electric field was found, as pointed out in Ref. 14. The correct relation is given by $\nabla'[\rho] = [\nabla'\rho] + \hat{\mathbf{R}}[\dot{\rho}]/c$ (see, for instance, Ref. 5). If we use the correct relation, Eq. (10) becomes

$$\mathbf{E}(\mathbf{x},t) = -\int d\mathbf{x}'\frac{\nabla'[\rho]}{R} + \int d\mathbf{x}'\frac{[\dot{\rho}]\hat{\mathbf{R}}}{cR} - \int d\mathbf{x}'\frac{[\dot{\mathbf{J}}]}{c^2R} \tag{11}$$

$$= \int d\mathbf{x}'\frac{[\rho]\hat{\mathbf{R}}}{R^2} + \int d\mathbf{x}'\frac{[\dot{\rho}]\hat{\mathbf{R}}}{cR} - \int d\mathbf{x}'\frac{[\dot{\mathbf{J}}]}{c^2R}. \tag{12}$$

In the last step we integrated by parts and discarded surface terms, because the charge distribution is localized. Equation (12) is one of the Jefimenko's equations.[4,5] Note the retarded character of the electric field. The first term on the right-hand side of Eq. (12) is the retarded Coulomb term.

As shown by Panofsky and Phillips,[7] there is an equivalent way of deriving Eq. (12) for the electric field which manifestly shows the transverse character of the radiation field. A discussion of this point can be found in Ref. 8; we will discuss this point in Sec. III.

To obtain the desired expression for the magnetic field, we use the retarded Green function in Eq. (8) to rewrite Eq. (6) as

$$\mathbf{B}(\mathbf{x},t) = \frac{1}{c}\int \frac{d\mathbf{x}'}{R}\int dt'\,\delta(t'-(t-R/c))\nabla'\times\mathbf{J}(\mathbf{x}',t')$$
$$= \frac{1}{c}\int \frac{d\mathbf{x}'}{R}[\nabla'\times\mathbf{J}]. \tag{13}$$

If we use the relation $\nabla'\times[\mathbf{J}] = [\nabla'\times\mathbf{J}] + (\hat{\mathbf{R}}/c)\times[\dot{\mathbf{J}}]$ (see Ref. 5), Eq. (13) takes the form

$$\mathbf{B}(\mathbf{x},t) = \frac{1}{c}\int \frac{d\mathbf{x}'}{R}\nabla'\times[\mathbf{J}] - \frac{1}{c^2}\int d\mathbf{x}'\frac{\hat{\mathbf{R}}\times[\dot{\mathbf{J}}]}{R}. \tag{14}$$

We integrate by parts and obtain

$$\mathbf{B}(\mathbf{x},t) = \frac{1}{c}\int d\mathbf{x}'\nabla'\times\left(\frac{[\mathbf{J}]}{R}\right) + \frac{1}{c}\int d\mathbf{x}'[\mathbf{J}]\times\nabla'\left(\frac{1}{R}\right) - \frac{1}{c^2}\int d\mathbf{x}'\frac{\hat{\mathbf{R}}\times[\dot{\mathbf{J}}]}{R}. \tag{15}$$

The first term on the right-hand side of Eq. (15) vanishes because the current distribution is localized in space. We use the relation $\nabla'(1/R) = -\hat{\mathbf{R}}/R^2$ to obtain

$$\mathbf{B}(\mathbf{x},t) = \int d\mathbf{x}'\left[\frac{[\mathbf{J}]\times\hat{\mathbf{R}}}{cR^2} + \frac{[\dot{\mathbf{J}}]\times\hat{\mathbf{R}}}{c^2R}\right]. \tag{16}$$

The first term on the right-hand side of Eq. (15) is the retarded Biot-Savart term; the transverse radiation field is contained in the last term. Equations (12) and (16) are the Jefimenko equations.

B. Fourier method

In Sec. II A we showed how to obtain Jefimenko's equations by a careful treatment of the derivatives of retarded quantities. This point is crucial—spatial derivatives cannot be commuted with retarding the functions, because the retarded function depends on the coordinates in its time argument. A simple way to circumvent this difficulty is to use Fourier transforms and factor out the time dependence of the functions so these subtleties are not encountered. We will show how to obtain the electric field and will leave the derivation of the magnetic field as an exercise for the interested reader (see, for example, Ref. 16).

We start with the electric field given by Eq. (9). The term involving the current density takes the form after integration over time,

$$-\int \frac{d\mathbf{x}'}{R}\int dt'\,\delta(t'-(t-R/c))\frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t'}(\mathbf{x}',t')$$
$$= -\int d\mathbf{x}'\frac{[\dot{\mathbf{J}}]}{c^2R}. \tag{17}$$

We introduce the Fourier transformation $\tilde{\rho}(\mathbf{x}',\omega)$ of $\rho(\mathbf{x}',t')$ as $\rho(\mathbf{x}',t') = \int \tilde{\rho}(\mathbf{x}',\omega)e^{-i\omega t'}d\omega$ and express the remaining contribution to the electric field in Eq. (10) as

$$-\int \frac{d\mathbf{x}'}{R}\int dt'\,\delta(t'-(t-R/c))\nabla'\rho(\mathbf{x}',t')$$
$$= -\int \frac{d\mathbf{x}'}{R}\int dt'\,\delta(t'-(t-R/c)) \times \int d\omega\nabla'\tilde{\rho}(\mathbf{x}',\omega)e^{-i\omega t'} \tag{18a}$$

$$= -\int d\omega e^{-i\omega t}\int d\mathbf{x}'\frac{e^{ikR}}{R}\nabla'\tilde{\rho}(\mathbf{x}',\omega) \tag{18b}$$

$$= -\int d\omega e^{-i\omega t}\int d\mathbf{x}'\left\{\nabla'\left(\frac{e^{ikR}}{R}\tilde{\rho}(\mathbf{x}',\omega)\right) - \tilde{\rho}(\mathbf{x}',\omega)\left(\frac{\hat{\mathbf{R}}}{R^2} - ik\frac{\hat{\mathbf{R}}}{R}\right)e^{ikR}\right\} \tag{18c}$$

$$= \int d\mathbf{x}' \frac{\hat{\mathbf{R}}}{R^2} \int d\omega \tilde{\rho}(\mathbf{x}',\omega) e^{-i\omega(t-R/c)}$$

$$+ \int d\mathbf{x}' \frac{\hat{\mathbf{R}}}{cR} \int d\omega \tilde{\rho}(\mathbf{x}',\omega)(-i\omega) e^{-i\omega(t-R/c)} \tag{18d}$$

$$= \int d\mathbf{x}' \frac{\hat{\mathbf{R}}[\rho]}{R^2} + \int d\mathbf{x}' \frac{\hat{\mathbf{R}}[\dot{\rho}]}{cR}. \tag{18e}$$

To obtain Eq. (18c) we integrated by parts; to obtain Eq. (18d) we discarded the surface term.

We combine Eq. (9) with Eqs. (17) and (18) and obtain the electric field generated by arbitrary (but localized) sources, namely, Eq. (12). On the right-hand side of Eq. (18a) it is obvious that $\boldsymbol{\nabla}'$ acts only on $\tilde{\rho}(\mathbf{x}',\omega)$. Hence, in Eq. (18b) there is no possibility of thinking that $\boldsymbol{\nabla}'$ also acts on the exponential e^{ikR}. The subtleties of dealing with retarded quantities are circumvented by the Fourier method.

C. Postponing the delta-function integration

When we are faced with integrals involving delta functions, we are usually tempted to do them first, because they are so easy. To derive Jefimenko's equation for the electric field given by Eq. (12), this procedure is not optimum. Let us start with Eq. (9):

$$\mathbf{E}(\mathbf{x},t) = -\int \frac{d\mathbf{x}'}{R} \int dt'\, \delta(t' - (t - R/c)) \times \left(\boldsymbol{\nabla}' \rho(\mathbf{x}',t') + \frac{1}{c^2} \frac{\partial \mathbf{J}}{\partial t'}(\mathbf{x}',t') \right). \tag{19}$$

Instead of first performing the time integration using the Dirac delta function, we integrate by parts on both terms on the right-hand side of Eq. (19) so that it takes the form

$$\mathbf{E}(\mathbf{x},t) = \int d\mathbf{x}' \int dt' \Bigg\{ \delta'(t' - (t - R/c)) \times \left(\frac{1}{Rc} \rho(\mathbf{x}',t') \boldsymbol{\nabla}' R + \frac{1}{Rc^2} \mathbf{J}(\mathbf{x}',t') \right) + \delta(t' - (t - R/c)) \left(\boldsymbol{\nabla}' \frac{1}{R} \right) \rho(\mathbf{x}',t') \Bigg\}. \tag{20}$$

If we use the relations $\boldsymbol{\nabla}' R = -\hat{\mathbf{R}}$ and $\boldsymbol{\nabla}' 1/R = \hat{\mathbf{R}}/R^2$, we obtain

$$\mathbf{E}(\mathbf{x},t) = \int d\mathbf{x}' \int dt' \Bigg\{ \delta'(t' - (t - R/c)) \left(-\frac{\hat{\mathbf{R}}}{Rc} \rho(\mathbf{x}',t') + \frac{1}{Rc^2} \mathbf{J}(\mathbf{x}',t') \right) + \delta(t' - (t - R/c)) \rho(\mathbf{x}',t') \frac{\hat{\mathbf{R}}}{R^2} \Bigg\}. \tag{21}$$

By using the properties of the Dirac delta function, we can easily perform the integration over t' to obtain Eq. (12). An analogous calculation provides an expression for the magnetic field.

III. MULTIPOLE RADIATION VIA JEFIMENKO'S EQUATIONS

The main purpose of this section is to add to the list of problems that can be handled directly with Jefimenko's equations by calculating the first multipole contributions to the radiation fields of an arbitrary localized source. We shall obtain the first three contributions, namely, the electric dipole, the magnetic dipole, and the electric quadrupole terms. Most textbooks treat this problem by calculating first the electromagnetic potentials.

Although Eq. (12) for the electric field gives the correct expression for the electric field of moving charges, it is preferable for our purposes to write it in an equivalent form as given in Refs. 7 and 8:

$$\mathbf{E}(\mathbf{x},t) = \int d\mathbf{x}' \frac{[\rho]\hat{\mathbf{R}}}{R^2} + \int d\mathbf{x}' \frac{([\mathbf{J}] \cdot \hat{\mathbf{R}})\hat{\mathbf{R}} + ([\mathbf{J}] \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}}}{cR^2} + \int d\mathbf{x}' \frac{([\dot{\mathbf{J}}] \times \hat{\mathbf{R}}) \times \hat{\mathbf{R}}}{c^2 R}. \tag{22}$$

We obtain Eq. (22) starting with Jefimenko's equation for the electric field in Eq. (12). Our derivation will follow the one in Ref. 8. Note that (the Einstein convention of implicit summation over repeated indices is assumed)

$$\boldsymbol{\nabla}' \cdot [\mathbf{J}] = \partial_i' J_i(\mathbf{x}',t')|_{t'=t-R/c} + \frac{\partial}{\partial t'} J_i(\mathbf{x}',t')|_{t'=t-R/c} \times \left(-\frac{1}{c} \partial_i' R \right) \tag{23a}$$

$$= [\boldsymbol{\nabla}' \cdot \mathbf{J}] + [\dot{J}_i] \left(\frac{X_i}{cR} \right) = -[\dot{\rho}] + [\dot{\mathbf{J}}] \cdot \frac{\hat{\mathbf{R}}}{c}, \tag{23b}$$

where $X_i \equiv x_i - x_i'$ and in the last step we used the continuity equation. From Eq. (23b) we have $[\dot{\rho}] = -\boldsymbol{\nabla}' \cdot [\mathbf{J}] + [\dot{\mathbf{J}}] \cdot \hat{\mathbf{R}}/c$, which can be substituted into the second term on the right-hand side of Eq. (12) to yield

$$\frac{1}{c} \int \frac{[\dot{\rho}]\hat{\mathbf{R}}}{R} d\mathbf{x}' = \frac{1}{c} \int \frac{(\boldsymbol{\nabla}' \cdot [\mathbf{J}])\hat{\mathbf{R}}}{R} d\mathbf{x}' + \frac{1}{c^2} \int \frac{([\dot{\mathbf{J}}] \cdot \hat{\mathbf{R}})\hat{\mathbf{R}}}{R} d\mathbf{x}'. \tag{24}$$

We show that the first term on the right-hand side of Eq. (24) is proportional to $1/R^2$, so that it does not contribute to the radiation field. Observe that

$$-\frac{1}{c} \int \frac{(\boldsymbol{\nabla}' \cdot [\mathbf{J}])\hat{\mathbf{R}}}{R} d\mathbf{x}' = -\frac{\hat{\mathbf{e}}_k}{c} \int (\partial_i' [J_i]) \frac{X_k}{R^2} d\mathbf{x}' \tag{25a}$$

$$= -\frac{\hat{\mathbf{e}}_k}{c} \int \partial_i' \left([J_i] \frac{X_k}{R^2} \right) d\mathbf{x}' + \frac{\hat{\mathbf{e}}_k}{c} \int [J_i] \partial_i' \left(\frac{X_k}{R^2} \right) d\mathbf{x}' \tag{25b}$$

$$=0+\frac{\hat{\mathbf{e}}_k}{c}\int [J_i]\left\{X_k\frac{2}{R^3}\frac{X_i}{R}-\frac{\delta_{ki}}{R^2}\right\}d\mathbf{x}' \tag{25c}$$

$$=\frac{1}{c}\int\frac{2([\mathbf{J}]\cdot\hat{\mathbf{R}})\hat{\mathbf{R}}-[\mathbf{J}]}{R^2}d\mathbf{x}' \tag{25d}$$

$$=\frac{1}{c}\int\frac{([\mathbf{J}]\cdot\hat{\mathbf{R}})\hat{\mathbf{R}}+([\mathbf{J}]\times\hat{\mathbf{R}})\times\hat{\mathbf{R}}}{R^2}d\mathbf{x}', \tag{25e}$$

where we have used the identity $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$, and the fact that the surface term appearing in Eq. (25b) vanishes because the sources are localized. We substitute Eq. (25) into Eq. (24) and insert the result into Eq. (12) to obtain

$$\mathbf{E}(\mathbf{x},t)=\int\frac{[\rho]\hat{\mathbf{R}}}{R^2}d\mathbf{x}'+\frac{1}{c}\int\frac{([\mathbf{J}]\cdot\hat{\mathbf{R}})\hat{\mathbf{R}}+([\mathbf{J}]\times\hat{\mathbf{R}})\times\hat{\mathbf{R}}}{R^2}d\mathbf{x}'+\frac{1}{c^2}\int\frac{([\dot{\mathbf{J}}]\cdot\hat{\mathbf{R}})\hat{\mathbf{R}}}{R}d\mathbf{x}'-\frac{1}{c^2}\int\frac{[\dot{\mathbf{J}}]}{R}d\mathbf{x}' \tag{26a}$$

$$=\int d\mathbf{x}'\frac{[\rho]\hat{\mathbf{R}}}{R^2}+\int d\mathbf{x}'\frac{([\mathbf{J}]\cdot\hat{\mathbf{R}})\hat{\mathbf{R}}+([\mathbf{J}]\times\hat{\mathbf{R}})\times\hat{\mathbf{R}}}{cR^2}+\int d\mathbf{x}'\frac{([\dot{\mathbf{J}}]\times\hat{\mathbf{R}})\times\hat{\mathbf{R}}}{c^2R}, \tag{26b}$$

which is Eq. (22). If we consider arbitrary time-varying sources at rest and use Eqs. (16) and (22), the (transverse) magnetic and electric radiation fields are given by

$$\mathbf{B}_{\mathrm{rad}}(\mathbf{x},t)=\frac{1}{c}\int d\mathbf{x}'\frac{[\dot{\mathbf{J}}]\times\hat{\mathbf{R}}}{Rc},\qquad \mathbf{E}_{\mathrm{rad}}(\mathbf{x},t)=\int d\mathbf{x}'\frac{([\dot{\mathbf{J}}]\times\hat{\mathbf{R}})\times\hat{\mathbf{R}}}{c^2R}. \tag{27}$$

It can be shown[17] that for time-varying sources in motion, Eq. (27) gives the radiation fields plus additional nonradiative terms of order $\mathcal{O}(1/R^2)$. For instance, if a Hertz dipole is accelerated, integration of Eq. (27) yields radiation fields plus nonradiative terms of order $\mathcal{O}(1/R^2)$ which are induced by the dipole motion (see Ref. 17 for details).

In the radiation zone we can write $\hat{\mathbf{R}}\simeq\hat{\mathbf{x}}$, $1/R\simeq 1/r$, and $R\simeq r-\hat{\mathbf{x}}\cdot\mathbf{x}'$, where we defined $r=|\mathbf{x}|$. If we substitute these approximations into Eq. (27), we obtain

$$\mathbf{B}_{\mathrm{rad}}(\mathbf{x},t)\simeq\frac{1}{c^2r}\int d\mathbf{x}'\dot{\mathbf{J}}\left(\mathbf{x}',t_0+\frac{\hat{\mathbf{x}}\cdot\mathbf{x}'}{c}\right)\times\hat{\mathbf{x}}, \tag{28}$$

$$\mathbf{E}_{\mathrm{rad}}(\mathbf{x},t)\simeq\frac{1}{c^2r}\int d\mathbf{x}'\left[\dot{\mathbf{J}}\left(\mathbf{x}',t_0+\frac{\hat{\mathbf{x}}\cdot\mathbf{x}'}{c}\right)\times\hat{\mathbf{x}}\right]\times\hat{\mathbf{x}}, \tag{29}$$

where $t_0=t-r/c$ is the retarded time of the origin. A comparison of Eqs. (28) and (37) leads to the relation

$$\mathbf{E}_{\mathrm{rad}}(\mathbf{x},t)=\mathbf{B}_{\mathrm{rad}}(\mathbf{x},t)\times\hat{\mathbf{x}}. \tag{30}$$

Now we are ready to calculate the first multipole contributions for the radiation fields. We need to calculate only one of the radiation fields because the other is readily obtained by Eq. (30), which also shows that the fields in the radiation zone are mutually orthogonal. We start by calculating the electric dipole term and then consider the next order contribution given by both the magnetic dipole and electric quadrupole terms.

A. The electric dipole contribution

The lowest order contribution to the radiation fields comes from the electric dipole term. For simplicity, we calculate the lowest order contribution to the radiation magnetic field, which we denote by $\mathbf{B}^{(1)}_{\mathrm{rad}}$,

$$\mathbf{B}^{(1)}_{\mathrm{rad}}(\mathbf{x},t)=\frac{1}{c^2r}\left\{\int d\mathbf{x}'\dot{\mathbf{J}}(\mathbf{x}',t_0)\right\}\times\hat{\mathbf{x}}. \tag{31}$$

We write the unit vectors of the Cartesian basis as $\hat{\mathbf{e}}_i=\boldsymbol{\nabla}'x'_i$, $(i=1,2,3)$, and write any vector $\mathbf{v}$ as $\mathbf{v}=\hat{\mathbf{e}}_iv_i=\hat{\mathbf{e}}_i(\mathbf{v}\cdot\hat{\mathbf{e}}_i)$. The integral in Eq. (31) can be expressed as

$$\int d\mathbf{x}'\dot{\mathbf{J}}(\mathbf{x}',t_0)=\hat{\mathbf{e}}_i\int d\mathbf{x}'\dot{\mathbf{J}}(\mathbf{x}',t_0)\cdot\hat{\mathbf{e}}_i=\hat{\mathbf{e}}_i\int d\mathbf{x}'\dot{\mathbf{J}}(\mathbf{x}',t_0)\cdot\nabla'x'_i \tag{32a}$$

$$=\hat{\mathbf{e}}_i\int d\mathbf{x}'\boldsymbol{\nabla}'\cdot(x'_i\dot{\mathbf{J}}(\mathbf{x}',t_0))-\hat{\mathbf{e}}_i\int d\mathbf{x}'x'_i\nabla'\cdot\dot{\mathbf{J}}(\mathbf{x}',t_0), \tag{32b}$$

where in the last step we integrated by parts. Because we are considering localized sources, the first integral on the right-hand side of Eq. (32a) vanishes (after the use of Gauss' theorem this integral is converted to a zero surface term). The remaining integral may be cast into a convenient form if we use the relation

$$\boldsymbol{\nabla}'\cdot\dot{\mathbf{J}}(\mathbf{x}',t_0)=-\frac{\partial^2\rho(\mathbf{x}',t_0)}{\partial t^2}, \tag{33}$$

which is a direct consequence of the continuity equation. To obtain Eq. (33) we used the relation $\partial\rho(\mathbf{x}',t_0)/\partial t=\partial\rho(\mathbf{x}',t_0)/\partial t_0$ (see Ref. 18, note 5). We substitute Eqs. (33) and (32a) into Eq. (31) and obtain

$$\mathbf{B}^{(1)}_{\mathrm{rad}}(\mathbf{x},t)=\frac{1}{c^2r}\frac{\partial^2}{\partial t^2}\left\{\overbrace{\int d\mathbf{x}'\rho(\mathbf{x}',t_0)\mathbf{x}'}^{\mathbf{p}(t_0)}\right\}\times\hat{\mathbf{x}}=\frac{\ddot{\mathbf{p}}(t_0)\times\hat{\mathbf{x}}}{c^2r}, \tag{34}$$

where $\mathbf{p}(t_0)$ is the electric dipole moment of the distribution at the retarded time t_0. Now it is clear why this first term is called the electric dipole term. The radiation electric field is readily obtained from Eq. (30)

$$\mathbf{E}_{\text{rad}}^{(1)}(\mathbf{x},t) = \frac{[\ddot{\mathbf{p}}(t_0) \times \hat{\mathbf{x}}] \times \hat{\mathbf{x}}}{c^2 r}. \tag{35}$$

Equations (31) and (35) are the radiation fields of the electric dipole term, that is, the first-order contribution to the multipole expansion. These expressions are valid for arbitrary but localized, sources in vacuum such as an oscillating electric dipole.

B. Next order contribution

To calculate the next order term we need to take into account the second term of the expansion

$$\dot{\mathbf{J}}\left(\mathbf{x}', t_0 + \frac{\hat{\mathbf{x}} \cdot \mathbf{x}'}{c}\right) \approx \dot{\mathbf{J}}(\mathbf{x}', t_0) + \frac{\hat{\mathbf{x}} \cdot \mathbf{x}'}{c} \ddot{\mathbf{J}}(\mathbf{x}', t_0). \tag{36}$$

We substitute Eq. (36) into Eq. (28) and identify the next order contribution to the radiation magnetic field, $\mathbf{B}_{\text{rad}}^{(2)}$, given by

$$\mathbf{B}_{\text{rad}}^{(2)}(\mathbf{x},t) = \frac{1}{c^3 r}\left(\int d\mathbf{x}' \ddot{\mathbf{J}}(\mathbf{x}', t_0)(\hat{\mathbf{x}} \cdot \mathbf{x}')\right) \times \hat{\mathbf{x}}. \tag{37}$$

For reasons that will become clear we will split the integral in Eq. (37) into antisymmetric and symmetric contributions under the exchange of $\mathbf{J}$ and $\mathbf{x}'$. This rearrangement will give rise to the magnetic dipole and electric quadrupole terms of the multipole expansion for the radiation fields.

If we write $\ddot{\mathbf{J}}(\mathbf{x}', t_0)(\hat{\mathbf{x}} \cdot \mathbf{x}')$ as $\frac{1}{2}\ddot{\mathbf{J}}(\mathbf{x}', t_0)(\hat{\mathbf{x}} \cdot \mathbf{x}') + \frac{1}{2}\ddot{\mathbf{J}}(\mathbf{x}', t_0) \times (\hat{\mathbf{x}} \cdot \mathbf{x}')$ and sum and subtract $\frac{1}{2}[\ddot{\mathbf{J}}(\mathbf{x}', t_0) \cdot \hat{\mathbf{x}}]\mathbf{x}'$ in the integrand of the right-hand side of Eq. (37), we obtain

$$\begin{aligned}
\mathbf{B}_{\text{rad}}^{(2)}(\mathbf{x},t) = \frac{1}{c^2 r}\Bigg\{ & \frac{1}{2c}\int d\mathbf{x}' [\ddot{\mathbf{J}}(\mathbf{x}', t_0)(\hat{\mathbf{x}} \cdot \mathbf{x}') \\
& - (\ddot{\mathbf{J}}(\mathbf{x}', t_0) \cdot \hat{\mathbf{x}})\mathbf{x}'] \Bigg\} \times \hat{\mathbf{x}} \\
& + \frac{1}{c^2 r}\Bigg\{ \frac{1}{2c}\int d\mathbf{x}' [\ddot{\mathbf{J}}(\mathbf{x}', t_0)(\hat{\mathbf{x}} \cdot \mathbf{x}') \\
& + (\ddot{\mathbf{J}}(\mathbf{x}', t_0) \cdot \hat{\mathbf{x}})\mathbf{x}'] \Bigg\} \times \hat{\mathbf{x}}.
\end{aligned} \tag{38}$$

For pedagogical reasons we shall treat the magnetic dipole and electric quadrupole cases separately.

Consider the antisymmetric term of Eq. (38). We denote this contribution by $\mathbf{B}_{\text{rad}}^{(2a)}$, use $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\mathbf{a}$, and write it in the following suggestive way:

$$\begin{aligned}
\mathbf{B}_{\text{rad}}^{(2a)}(\mathbf{x},t) = \frac{1}{c^2 r}\frac{1}{2c}\int & d\mathbf{x}' [\ddot{\mathbf{J}}(\mathbf{x}', t_0)\hat{\mathbf{x}} \cdot \mathbf{x}' \\
& - (\ddot{\mathbf{J}}(\mathbf{x}', t_0) \cdot \hat{\mathbf{x}})\mathbf{x}'] \times \hat{\mathbf{x}}
\end{aligned} \tag{39a}$$

$$= \frac{1}{c^2 r}\left\{ \underbrace{\left(\int d\mathbf{x}' \frac{\mathbf{x}' \times \ddot{\mathbf{J}}(\mathbf{x}', t_0)}{2c}\right)}_{\ddot{\mathbf{m}}(t_0)} \times \hat{\mathbf{x}} \right\} \times \hat{\mathbf{x}}, \tag{39b}$$

where we have identified the second time derivative of the magnetic dipole moment of the charge distribution at the retarded time as $\ddot{\mathbf{m}}(t_0)$. Hence, the antisymmetric contribution for the radiation magnetic field is given by

$$\mathbf{B}_{\text{rad}}^{(2a)}(\mathbf{x},t) = \frac{[\ddot{\mathbf{m}}(t_0) \times \hat{\mathbf{x}}] \times \hat{\mathbf{x}}}{c^2 r}. \tag{40}$$

The appearance of the magnetic dipole moment of the distribution justifies the name given to this contribution. The corresponding radiation electric field is readily given by

$$\mathbf{E}_{\text{rad}}^{(2a)}(\mathbf{x},t) = \frac{\hat{\mathbf{x}} \times \ddot{\mathbf{m}}(t_0)}{c^2 r}. \tag{41}$$

Expressions (40) and (41) are valid for an arbitrary, but localized, time-varying source at rest in vacuum such as an oscillating magnetic dipole.

We denote the symmetric term of Eq. (38) as $\mathbf{B}_{\text{rad}}^{(2s)}$. We have

$$\begin{aligned}
\mathbf{B}_{\text{rad}}^{(2s)}(\mathbf{x},t) = \frac{1}{c^2 r}\frac{1}{2c}\int & d\mathbf{x}' \{\ddot{\mathbf{J}}(\mathbf{x}', t_0)(\hat{\mathbf{x}} \cdot \mathbf{x}') \\
& + (\ddot{\mathbf{J}}(\mathbf{x}', t_0) \cdot \hat{\mathbf{x}})\mathbf{x}'\} \times \hat{\mathbf{x}}.
\end{aligned} \tag{42}$$

We manipulate the first integral on the right-hand side of Eq. (42) in the same way as before. We have

$$\int d\mathbf{x}' \ddot{J}(\mathbf{x}', t_0)(\hat{\mathbf{x}} \cdot \mathbf{x}') = \hat{\mathbf{e}}_i \int d\mathbf{x}' (\hat{\mathbf{x}} \cdot \mathbf{x}')\ddot{\mathbf{J}}(\mathbf{x}', t_0) \cdot \hat{\mathbf{e}}_i \tag{43a}$$

$$= \hat{\mathbf{e}}_i \int d\mathbf{x}' (\hat{\mathbf{x}} \cdot \mathbf{x}')\ddot{\mathbf{J}}(\mathbf{x}', t_0) \cdot \nabla' x_i'. \tag{43b}$$

If we integrate by parts and remember that the surface term vanishes, we obtain

$$\begin{aligned}
\int & d\mathbf{x}' \ddot{J}(\mathbf{x}', t_0)(\hat{\mathbf{x}} \cdot \mathbf{x}') \\
& = -\hat{\mathbf{e}}_i \int d\mathbf{x}' x_i' \boldsymbol{\nabla}' \cdot [(\hat{\mathbf{x}} \cdot \mathbf{x}')\ddot{\mathbf{J}}(\mathbf{x}', t_0)]
\end{aligned} \tag{44a}$$

$$\begin{aligned}
= \hat{\mathbf{e}}_i \int & d\mathbf{x}' x_i' [(\nabla'(\hat{\mathbf{x}} \cdot \mathbf{x}')) \cdot \ddot{\mathbf{J}}(\mathbf{x}', t_0) \\
& + (\hat{\mathbf{x}} \cdot \mathbf{x}')\nabla' \cdot \ddot{\mathbf{J}}(\mathbf{x}', t_0)].
\end{aligned} \tag{44b}$$

We use $\boldsymbol{\nabla}' \cdot \ddot{\mathbf{J}}(\mathbf{x}', t_0) = -\partial^3 \rho(\mathbf{x}', t_0)/\partial t^3$, the relation[18] $\partial\rho(\mathbf{x}', t_0)/\partial t = \partial\rho(\mathbf{x}', t_0)/\partial t_0)$, and $\boldsymbol{\nabla}'(\hat{\mathbf{x}} \cdot \mathbf{x}') = \hat{\mathbf{x}}$ to obtain

$$\begin{aligned}
\int d\mathbf{x}' \ddot{J}(\mathbf{x}', t_0)(\hat{\mathbf{x}} \cdot \mathbf{x}') = & -\hat{\mathbf{e}}_i \int d\mathbf{x}' \hat{\mathbf{x}} \cdot \ddot{\mathbf{J}}(\mathbf{x}', t_0) x_i' \\
& + \hat{\mathbf{e}}_i \frac{\partial^3}{\partial t^3} \int d\mathbf{x}' (\hat{\mathbf{x}} \cdot \mathbf{x}') x_i' \rho(\mathbf{x}', t_0),
\end{aligned} \tag{45}$$

which implies

$$\int d\mathbf{x}'[\ddot{\mathbf{J}}(\mathbf{x}',t_0)(\hat{\mathbf{x}}\cdot\mathbf{x}')+(\ddot{\mathbf{J}}(\mathbf{x}',t_0)\cdot\hat{\mathbf{x}})\mathbf{x}']$$

$$=\frac{\partial^3}{\partial t^3}\int d\mathbf{x}'(\hat{\mathbf{x}}\cdot\mathbf{x}')\mathbf{x}'\rho(\mathbf{x}',t_0). \tag{46}$$

The left-hand side of Eq. (46) is the integral in Eq. (42). We write the factor $1/2$ in Eq. (42) as $3/6$ and obtain

$$\mathbf{B}_{\mathrm{rad}}^{(2s)}(\mathbf{x},t)=\frac{1}{6c^3r}\left\{\frac{\partial^3}{\partial t^3}\int d\mathbf{x}'(3\hat{\mathbf{x}}\cdot\mathbf{x}')\mathbf{x}'\rho(\mathbf{x}',t_0)\right\}\times\hat{\mathbf{x}} \tag{47a}$$

$$=\frac{1}{6c^3r}\left\{\frac{\partial^3}{\partial t^3}\int d\mathbf{x}'(3(\hat{\mathbf{x}}\cdot\mathbf{x}')\mathbf{x}'+r'^2\hat{\mathbf{x}})\times\rho(\mathbf{x}',t_0)\right\}\times\hat{\mathbf{x}}, \tag{47b}$$

where in the last step we included in the integrand the term $r'^2\hat{\mathbf{x}}\rho(\mathbf{x}',t_0)$, where $r'=|\mathbf{r}'|$, which gives a vanishing contribution to the result because $\hat{\mathbf{x}}\times\hat{\mathbf{x}}=0$. Now, we define the transformation $\mathbf{Q}$ such that

$$\mathbf{Q}(\boldsymbol{\xi},t)=\int d\mathbf{x}'[3(\boldsymbol{\xi}\cdot\mathbf{x}')\mathbf{x}'+r'^2\boldsymbol{\xi}]\rho(\mathbf{x}',t). \tag{48}$$

This transformation takes a vector and an instant of time $(\boldsymbol{\xi},t)$ and transforms it into the vector $\mathbf{Q}(\boldsymbol{\xi},t)$, given by Eq. (48). If we use the definition (48), Eq. (47a) for the antisymmetric contribution for $\mathbf{B}_{\mathrm{rad}}^{(2)}$ becomes

$$\mathbf{B}_{\mathrm{rad}}^{(2s)}(\mathbf{x},t)=\frac{1}{6c^3r}\dddot{\mathbf{Q}}(\hat{\mathbf{x}},t_0)\times\hat{\mathbf{x}}. \tag{49}$$

The corresponding symmetric contribution for the radiating electric field is obtained from Eq. (30):

$$\mathbf{E}_{\mathrm{rad}}^{(2s)}(\mathbf{x},t)=\frac{1}{6c^3r}[\dddot{\mathbf{Q}}(\hat{\mathbf{x}},t_0)\times\hat{\mathbf{x}}]\times\hat{\mathbf{x}}. \tag{50}$$

In order to interpret the above term, let us define a linear operator $\mathbf{Q}^t$, for a fixed instant t, by $\mathbf{Q}^t(\boldsymbol{\xi})\equiv\mathbf{Q}(\boldsymbol{\xi},t)$. Note that $\mathbf{Q}^t$ is a linear operator so that $\mathbf{Q}^t(\alpha_1\boldsymbol{\xi}_1+\alpha_2\boldsymbol{\xi}_2)=\alpha_1\mathbf{Q}^t(\boldsymbol{\xi}_1)+\alpha_2\mathbf{Q}^t(\boldsymbol{\xi}_2)$, for α_1, $\alpha_2\in\mathcal{R}$. The linear operator $\mathbf{Q}^t$ is called the electric quadrupole operator. Because $\mathbf{Q}^t(\hat{\mathbf{e}}_i)$ is a vector in $\mathcal{R}^3$, we can write it as a linear combination of the basis vectors, namely

$$\mathbf{Q}^t(\hat{\mathbf{e}}_i)=\sum_{j=1}^{3}\mathbf{Q}^t_{ji}\hat{\mathbf{e}}_j \quad (i=1,2,3). \tag{51}$$

The coefficients $\mathbf{Q}^t_{ji}$ are the Cartesian elements of the electric quadrupole tensor (a second rank tensor) of the distribution at instant t. That is why we interpret Eqs. (49) and (50) as the electric quadrupole contribution for the radiation fields.

The electric quadrupole radiation fields given by Eqs. (49) and (50), together with the magnetic dipole radiation fields given by Eqs. (40) and (41), are the first corrections to the leading order term given by Eqs. (34) and (35). In this sense, we can write the first multipole contributions to the multipole expansion for the radiation fields of a completely arbitrary, but localized, time-varying source at rest in vacuum as

$$\mathbf{B}_{\mathrm{rad}}(\mathbf{x},t)=\frac{1}{c^2r}\left\{\ddot{\mathbf{p}}(t_0)\times\hat{\mathbf{x}}+[\ddot{\mathbf{m}}(t_0)\times\hat{\mathbf{x}}]\times\hat{\mathbf{x}}+\frac{1}{3c}\dddot{\mathbf{Q}}(\hat{\mathbf{x}},t_0)\times\hat{\mathbf{x}}+\cdots\right\}, \tag{52}$$

$$\mathbf{E}_{\mathrm{rad}}(\mathbf{x},t)=\mathbf{B}_{\mathrm{rad}}(\mathbf{x},t)\times\hat{\mathbf{x}}. \tag{53}$$

With these radiation fields, we can calculate the corresponding Poynting vector and the power radiated by the source.

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