Rutherford scattering with radiation damping

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We study the effect of radiation damping on the classical scattering of charged particles. By using a perturbation method based on the Runge–Lenz vector, we calculate radiative corrections to the Rutherford cross section and deflection function. Energy and angular momentum losses are obtained by the same method.
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I. INTRODUCTION

The reaction of a classical point charge to its own radiation was first discussed by Lorentz and Abraham more than one hundred years ago and is still a source of controversy and fascination.[1–4] Radiation damping considerably complicates the equations of motion of charged particles, and for many important problems, such as Rutherford scattering, only numerical calculations of the trajectories are available.[5,6]

In this paper we study the effect of radiation reaction on the classical two-body scattering of charged particles. Following Landau and Lifshitz,[2] we expand the electromagnetic force in powers of c^{-1} (c is the speed of light) to order c^{-3} where radiation damping appears. We then use a perturbation technique based on the Runge–Lenz vector[7] to calculate the radiation damping corrections to Rutherford's deflection function and scattering cross section. The corresponding angular momentum and energy losses are also calculated. Our results show that radiation damping effects are limited if the colliding charges have the same sign. For opposite sign charges radiative corrections can be very large and depend strongly on the scattering angle.

This paper is organized as follows. In Sec. II we obtain the radiation damping force on a system of charged particles starting from an expansion of the electromagnetic field in powers of c^{-1}. The equations of motion for a two-body system with radiation reaction are discussed in Sec. III. In Sec. IV we use the Runge–Lenz vector to calculate the radiation effects on classical Rutherford scattering. Some final observations are made in Sec. V.

II. THE RADIATION DAMPING FORCE

For completeness we reproduce the derivation of the radiation damping force given by Landau and Lifshitz.[2] We start from the electromagnetic potentials $\phi(\mathbf{r},t)$ and $\mathcal{A}(\mathbf{r},t)$ created by the charge and current densities $\rho(\mathbf{r},t)$ and $\mathbf{J}(\mathbf{r},t)$,

$$\phi(\mathbf{r},t) = \int \frac{\rho(\mathbf{r}',t_R)}{R} d^3\mathbf{r}', \tag{1}$$

$$\mathcal{A}(\mathbf{r},t) = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{r}',t_R)}{R} d^3\mathbf{r}', \tag{2}$$

where $R=|\mathbf{r}-\mathbf{r}'|$ and $t_R=t-R/c$ is the retarded time. The electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, are given by

$$\mathbf{E} = -\nabla\phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial \mathcal{A}(\mathbf{r},t)}{\partial t}, \quad \mathbf{B} = \nabla\times\mathcal{A}(\mathbf{r},t). \tag{3}$$

We want to calculate the electromagnetic force on a charge q,

$$\mathbf{F} = q\mathbf{E} + \frac{q}{c}\mathbf{v}\times\mathbf{B}, \tag{4}$$

as a series in powers of $1/c$. If we expand $\rho(\mathbf{r}',t_R)$ and $\mathbf{J}(\mathbf{r}',t_R)$ about $t_R=t$, we obtain

$$\rho(\mathbf{r}',t_R) = \rho(\mathbf{r}',t) + \frac{\partial\rho(\mathbf{r}',t)}{\partial t}\left(-\frac{R}{c}\right) + \frac{1}{2}\frac{\partial^2\rho(\mathbf{r}',t)}{\partial t^2}\left(-\frac{R}{c}\right)^2 + \frac{1}{6}\frac{\partial^3\rho(\mathbf{r}',t)}{\partial t^3}\left(-\frac{R}{c}\right)^3 + \mathcal{O}(c^{-4}), \tag{5}$$

$$\mathbf{J}(\mathbf{r}',t_R) = \mathbf{J}(\mathbf{r}',t) + \frac{\partial\mathbf{J}(\mathbf{r}',t)}{\partial t}\left(-\frac{R}{c}\right) + \mathcal{O}(c^{-2}). \tag{6}$$

We substitute these expansions into Eqs. (1) and (2) and use the charge conservation relation,

$$\frac{\partial}{\partial t}\int \rho(\mathbf{r}',t)d^3\mathbf{r}' = 0, \tag{7}$$

and obtain

$$\phi(\mathbf{r},t) = \int \frac{\rho(\mathbf{r}',t)}{R}d^3\mathbf{r}' + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho(\mathbf{r}',t)d^3\mathbf{r}' - \frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int R^2\rho(\mathbf{r}',t)d^3\mathbf{r}' + \mathcal{O}(c^{-4}), \tag{8}$$

$$\frac{1}{c}\mathcal{A}(\mathbf{r},t) = \frac{1}{c^2}\int \frac{\mathbf{J}(\mathbf{r}',t)}{R}d^3\mathbf{r}' - \frac{1}{c^3}\frac{\partial}{\partial t}\int \mathbf{J}(\mathbf{r}',t)d^3\mathbf{r}' + \mathcal{O}(c^{-4}). \tag{9}$$

If we apply the gauge transformation

$$\phi(\mathbf{r},t) \to \phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial\chi(\mathbf{r},t)}{\partial t}, \tag{10}$$

$$\mathcal{A}(\mathbf{r},t) \to \mathcal{A}(\mathbf{r},t) + \nabla\chi(\mathbf{r},t), \tag{11}$$

where

$$\chi(\mathbf{r},t)=\frac{1}{2c}\frac{\partial}{\partial t}\int R\rho(\mathbf{r},t)d^3\mathbf{r}'-\frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\int R^2\rho(\mathbf{r},t)d^3\mathbf{r}', \tag{12}$$

we can rewrite Eqs. (8) and (9) as

$$\phi(\mathbf{r},t)=\int\frac{\rho(\mathbf{r}',t)}{R}d^3\mathbf{r}'+\mathcal{O}(c^{-4}), \tag{13}$$

$$\begin{aligned}\frac{1}{c}\mathcal{A}(\mathbf{r},t)=&\frac{1}{c^2}\int\frac{\mathbf{J}(\mathbf{r}',t)}{R}d^3\mathbf{r}'+\frac{1}{2c^2}\frac{\partial}{\partial t}\int\frac{\mathbf{r}}{R}\rho(\mathbf{r}',t)d^3\mathbf{r}'\\&-\frac{1}{c^3}\frac{\partial}{\partial t}\int\mathbf{J}(\mathbf{r}',t)d^3\mathbf{r}'\\&-\frac{1}{3c^3}\frac{\partial^2}{\partial t^2}\int\mathbf{r}\rho(\mathbf{r}',t)d^3\mathbf{r}'+\mathcal{O}(c^{-4}).\end{aligned} \tag{14}$$

For a set of point charges q_k with positions $\mathbf{r}_k(t)$ and velocities $\mathbf{v}_k(t)$ we have

$$\rho(\mathbf{r},t)=\sum_k q_k\delta(\mathbf{r}-\mathbf{r}_k(t)), \tag{15}$$

$$\mathbf{J}(\mathbf{r},t)=\sum_k q_k\mathbf{v}_k(t)\delta(\mathbf{r}-\mathbf{r}_k(t)), \tag{16}$$

and the potentials become

$$\phi(\mathbf{r},t)=\sum_k\frac{q_k}{R_k(t)}+\mathcal{O}(c^{-4}), \tag{17}$$

$$\begin{aligned}\frac{1}{c}\mathcal{A}(\mathbf{r},t)=&\frac{1}{c^2}\sum_k\frac{q_k\mathbf{v}_k(t)}{R_k(t)}+\frac{1}{2c^2}\frac{d}{dt}\sum_k\frac{\mathbf{R}_k(t)}{R_k(t)}q_k\\&-\frac{1}{c^3}\frac{d}{dt}\sum_k q_k\mathbf{v}_k(t)-\frac{1}{3c^3}\frac{d^2}{dt^2}\sum_k\mathbf{R}_k(t)q_k\\&+\mathcal{O}(c^{-4}),\end{aligned} \tag{18}$$

with $\mathbf{R}_k(t)=\mathbf{r}-\mathbf{r}_k(t)$. We calculate the time derivatives in Eq. (18) and obtain

$$\begin{aligned}\frac{1}{c}\mathcal{A}(\mathbf{r},t)=&\frac{1}{2c^2}\sum_k\left[\frac{q_k\mathbf{v}_k(t)}{R_k(t)}+\frac{q_k\mathbf{R}_k(t)\cdot\mathbf{v}_k(t)}{R_k^3(t)}\mathbf{R}_k(t)\right]\\&-\frac{2}{3c^3}\sum_k q_k\mathbf{a}_k(t)+\mathcal{O}(c^{-4}),\end{aligned} \tag{19}$$

where $\mathbf{a}_k(t)$ is the acceleration of particle k.

The potential ϕ in Eq. (17) accounts for the Coulomb interaction. The first term in Eq. (19), which is order $1/c^2$, introduces magnetic and retardation effects, and can be used to set up the Darwin Lagrangian.[2] The last term in Eq. (19), which is order $1/c^3$, gives rise to the radiation damping electric field

$$\mathbf{E}_{\rm rad}=\frac{2}{3c^3}\sum_k q_k\frac{d\mathbf{a}_k}{dt}, \tag{20}$$

and a zero magnetic field ($\mathcal{A}$ is independent of $\mathbf{r}$ in this order).

It is interesting to see that the radiation reaction field is uniform. If we introduce the electric dipole of the system, $\mathbf{D}=\Sigma_k q_k\mathbf{r}_k$, the radiation damping field can be written as

$$\mathbf{E}_{\rm rad}=\frac{2}{3c^3}\frac{d^3\mathbf{D}}{dt^3}, \tag{21}$$

showing that $\mathbf{E}_{\rm rad}$ represents the reaction to the electric dipole radiation emitted by the entire system.

The radiation damping force on charge q_i becomes

$$\mathbf{F}_{\rm rad}^{(i)}=q_i\mathbf{E}_{\rm rad}=\frac{2}{3c^3}\sum_k q_iq_k\frac{d\mathbf{a}_k}{dt}. \tag{22}$$

Note from Eq. (22) that radiation reaction is not just a self-force—it receives contributions from every particle in the system. Only for a single accelerating charge q does the radiation damping force reduce to the Abraham–Lorentz self-interaction

$$\mathbf{F}_{\rm rad}=\frac{2}{3}\frac{q^2}{c^3}\frac{d\mathbf{a}}{dt}. \tag{23}$$

III. TWO-BODY MOTION WITH RADIATION DAMPING

Let us consider a system of two charged particles. We take radiation damping into account so that their equations of motion are

$$\frac{d^2\mathbf{r}_1}{dt^2}=\frac{q_1q_2}{m_1}\frac{\mathbf{r}}{r^3}+\frac{2}{3c^3}\frac{q_1}{m_1}\frac{d}{dt}(q_1\mathbf{a}_1+q_2\mathbf{a}_2), \tag{24a}$$

$$\frac{d^2\mathbf{r}_2}{dt^2}=-\frac{q_1q_2}{m_2}\frac{\mathbf{r}}{r^3}+\frac{2}{3c^3}\frac{q_2}{m_2}\frac{d}{dt}(q_1\mathbf{a}_1+q_2\mathbf{a}_2), \tag{24b}$$

where $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ and m_i is the mass of particle i. In Eq. (24) we have discarded the c^{-2} terms that account for the variation of mass with velocity and the Darwin magnetic and retardation effects. These terms do not interfere with our treatment of radiation damping; their effect on Rutherford scattering is discussed in Refs. 7 and 8.

If we subtract Eq. (24b) from (24a), we find

$$\frac{d^2\mathbf{r}}{dt^2}=\frac{q_1q_2}{\mu}\frac{\mathbf{r}}{r^3}+\frac{2}{3c^3}\left(\frac{q_1}{m_1}-\frac{q_2}{m_2}\right)\frac{d}{dt}(q_1\mathbf{a}_1+q_2\mathbf{a}_2), \tag{25}$$

where $\mu=m_1m_2/(m_1+m_2)$ is the reduced mass. We keep only the lowest order (c^0) terms in Eqs. (24) and (25) and find that

$$q_1\mathbf{a}_1+q_2\mathbf{a}_2=\mu\left(\frac{q_1}{m_1}-\frac{q_2}{m_2}\right)\frac{d^2\mathbf{r}}{dt^2}. \tag{26}$$

The substitution of this result into Eq. (25) gives

$$\frac{d^2\mathbf{r}}{dt^2}=\frac{q_1q_2}{\mu}\frac{\mathbf{r}}{r^3}+\frac{2\tilde{q}^2}{3\mu c^3}\frac{d^3\mathbf{r}}{dt^3}, \tag{27}$$

where

$$\tilde{q}=\mu\left(\frac{q_1}{m_1}-\frac{q_2}{m_2}\right). \tag{28}$$

In the fixed target limit, $m_2\to\infty$, Eq. (27) becomes the nonrelativistic Lorentz–Abraham equation of motion. Note that two-body recoil effects appear in Eq. (27) not only through the reduced mass μ, but also via the effective charge $\tilde{q}$. In particular, if $q_1/m_1=q_2/m_2$, we have $\tilde{q}=0$, and there is no radiation reaction even though both particles are accelerating. In this case the electric dipole vanishes (in the center of mass frame) and there is no electric dipole radiation from the system.

IV. RADIATIVE CORRECTIONS TO RUTHERFORD SCATTERING

In the absence of perturbations Rutherford scattering conserves the total energy $E=\mu v^2/2+q_1q_2/r$, the angular momentum $\mathbf{L}=\mu\mathbf{r}\times\mathbf{v}$, and the Runge–Lenz vector[9]

$$\mathbf{A}=\mathbf{v}\times\mathbf{L}+q_1q_2 r. \tag{29}$$

Here, $\mathbf{v}=d\mathbf{r}/dt$ is the relative velocity and $\hat{\mathbf{r}}=\mathbf{r}/r$ is the radial unit vector. These conserved quantities are not independent; we can show that $\mathbf{A}\cdot\mathbf{L}=0$, and

$$A^2=2EL^2/\mu+(q_1q_2)^2=(v_0L)^2+(q_1q_2)^2, \tag{30}$$

where v_0 is the initial (asymptotic) velocity. If we take the scalar product $\mathbf{r}\cdot\mathbf{A}$, we find the Rutherford scattering orbit

$$r(\varphi)=\frac{L^2/\mu}{A\cos\varphi-q_1q_2}, \tag{31}$$

where φ is the angle between $\mathbf{r}$ and $\mathbf{A}$. During the collision φ changes from $-\varphi_0$ to φ_0, where

$$\varphi_0=\cos^{-1}(q_1q_2/A)=\sin^{-1}(v_0L/A)=\tan^{-1}(v_0L/q_1q_2). \tag{32}$$

The scattering angle is $\theta=\pi-2\varphi_0$, and from Eq. (32) we obtain the Rutherford deflection function

$$\theta(L)=2\tan^{-1}(q_1q_2/v_0L). \tag{33}$$

For charges of the same sign the scattering angle is positive, and for opposite charges θ is negative (we take L and v_0 as always positive).

When radiation damping is considered, E, $\mathbf{L}$, and $\mathbf{A}$ are no longer conserved. In particular, from Eq. (27) we find that the Runge–Lenz vector changes at the rate

$$\frac{d\mathbf{A}}{dt}=\frac{2\tilde{q}^2}{3c^3}\left[\frac{1}{\mu}\frac{d^3\mathbf{r}}{dt^3}\times\mathbf{L}+\mathbf{v}\times\left(\mathbf{r}\times\frac{d^3\mathbf{r}}{dt^3}\right)\right]. \tag{34}$$

The total change of $\mathbf{A}$ during the collision is

$$\delta\mathbf{A}=\frac{2\tilde{q}^2}{3c^3}\int_{-\infty}^{\infty}dt\left[\frac{1}{\mu}\frac{d^3\mathbf{r}}{dt^3}\times\mathbf{L}+\mathbf{v}\times\left(\mathbf{r}\times\frac{d^3\mathbf{r}}{dt^3}\right)\right]. \tag{35}$$

The change of the Runge–Lenz vector is of order c^{-3}. We keep the same order of approximation and substitute into the integrand of Eq. (35) the results of unperturbed Rutherford scattering and obtain

$$\delta\mathbf{A}=\frac{2\tilde{q}^2}{3c^3}\frac{q_1q_2}{\mu^2}\int_{-\infty}^{\infty}dt\frac{\mathbf{A}-q_1q_2\hat{\mathbf{r}}}{r^3}, \tag{36}$$

which is further simplified by a change of variable from time t to angle φ. To order c^{-3} we have

$$dt=\frac{\mu r^2}{L}d\varphi, \tag{37}$$

so that

$$\delta\mathbf{A}=\frac{2\tilde{q}^2}{3c^3}\frac{q_1q_2}{\mu L}\int_{-\varphi_0}^{\varphi_0}d\varphi\frac{\mathbf{A}-q_1q_2\hat{\mathbf{r}}}{r}. \tag{38}$$

We substitute $r(\varphi)$ from Eq. (31) into Eq. (38) so that the integral reduces to

$$\delta\mathbf{A}=\frac{2\tilde{q}^2}{3c^3}\frac{q_1q_2}{L^3A}\mathbf{A}\int_{-\varphi_0}^{\varphi_0}d\varphi(A\cos\varphi-q_1q_2)\times(A-q_1q_2\cos\varphi), \tag{39}$$

which is easily calculated to be

$$\delta\mathbf{A}=\frac{2\tilde{q}^2}{3c^3}\frac{q_1q_2}{L^3A}\mathbf{A}[2(A^2+(q_1q_2)^2)\sin\varphi_0-q_1q_2A\sin\varphi_0\cos\varphi_0-3q_1q_2A\varphi_0]. \tag{40}$$

If we use Eq. (32), $\delta\mathbf{A}$ can be written as

$$\delta\mathbf{A}=\frac{2\tilde{q}^2}{3c^3}\frac{q_1q_2v_0}{L^2}\left[2+\frac{1}{1+(v_0L/q_1q_2)^2}-3\frac{q_1q_2}{v_0L}\tan^{-1}(v_0L/q_1q_2)\right]\mathbf{A}. \tag{41}$$

We see that radiation damping does not modify the direction of the Runge–Lenz vector, only its modulus. This modification changes the asymptotic angle $\varphi_0=\cos^{-1}(q_1q_2/A)$ by

$$\delta\varphi_0=\frac{q_1q_2}{v_0L}\frac{\delta A}{A}. \tag{42}$$

Accordingly, the scattering angle θ changes by (see Ref. 7)

$$\delta\theta=-\delta\varphi_0, \tag{43}$$

and the deflection function is

$$\theta(L)=2\tan^{-1}(q_1q_2/v_0L)+\delta\theta(L), \tag{44}$$

where the first term is the Rutherford relation and the radiation damping correction is

$$\delta\theta(L)=-\frac{2\tilde{q}^2}{3c^3}\frac{(q_1q_2)^2}{L^3}\left[2+\frac{1}{1+(v_0L/q_1q_2)^2}-3\frac{q_1q_2}{v_0L}\tan^{-1}(v_0L/q_1q_2)\right]. \tag{45}$$

From Eqs. (44) and (45) we can calculate $L(\theta)$. To order c^{-3} the result is

$$L(\theta)=\frac{q_1q_2}{v_0}\cot(\theta/2)\left[1+\frac{\tilde{q}^2}{q_1q_2}\left(\frac{v_0}{c}\right)^3\lambda(\theta)\right], \tag{46}$$

where

$$\lambda(\theta)=\frac{1}{6}\frac{\sin^3(\theta/2)}{\cos^5(\theta/2)}[(5-\cos\theta)\cot(\theta/2)-3(\pi-\theta)]. \tag{47}$$

A plot of $\lambda(\theta)$ is shown in Fig. 1. We see that the radiative correction is limited if the Coulomb force is repulsive, and is strongly divergent for backscattering ($\theta\to-\pi$) in an attractive Coulomb field.

The scattering cross section is calculated from the deflection function as

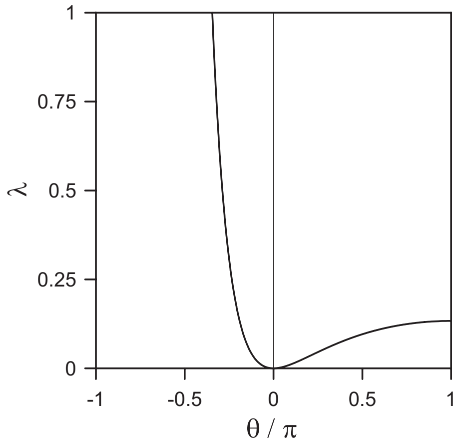

Fig. 1. Angular dependence of the radiative correction to Rutherford's deflection function. Positive (negative) angles correspond to the scattering of like (unlike) charges.

$$\frac{d\sigma}{d\Omega}=\frac{1}{p^2}\left|\frac{L}{\sin\theta}\frac{dL}{d\theta}\right|, \tag{48}$$

where $p=\mu v_0$ is the initial momentum. With Eqs. (46) and (47) we find

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma_R}{d\Omega}\left[1+\frac{\tilde{q}^2}{q_1q_2}\left(\frac{v_0}{c}\right)^3\xi(\theta)\right], \tag{49}$$

where

$$\frac{d\sigma_R}{d\Omega}=\left(\frac{q_1q_2}{2\mu v_0^2}\right)^2\frac{1}{\sin^4(\theta/2)} \tag{50}$$

is the nonrelativistic Rutherford cross section, and

$$\xi(\theta)=\frac{1}{2}\frac{\sin^3(\theta/2)}{\cos^5(\theta/2)}[(\pi-\theta)(2-\cos\theta)-3\sin\theta]. \tag{51}$$

The function $\xi(\theta)$ is shown in Fig. 2. At large angles, close to backscattering, $\xi(\theta)$ has the limiting behavior

$$\xi(\theta)\sim\frac{4}{15}-\frac{2}{35}(\theta-\pi)^2+\cdots\quad(\theta\to\pi), \tag{52a}$$

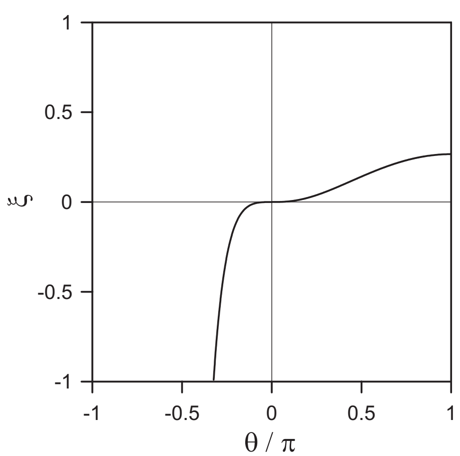

Fig. 2. Angular dependence of the radiative correction to the Rutherford cross section. The positive/negative angles are the same as in Fig. 1.

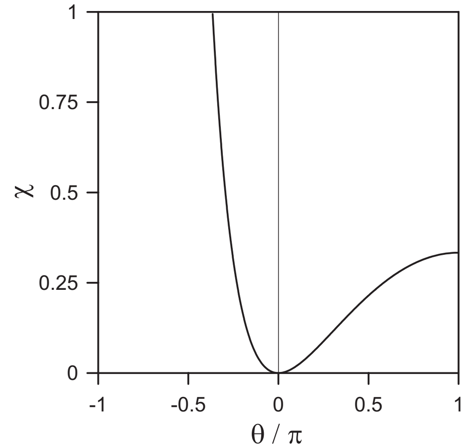

Fig. 3. Angular dependence of the change in angular momentum. Positive/negative angles are the same as in Fig. 1.

$$\xi(\theta)\sim-96\pi(\theta+\pi)^{-5}+\cdots\quad(\theta\to-\pi). \tag{52b}$$

The angular momentum loss (or gain) can be calculated with similar methods. With radiation damping, the time derivative of $\mathbf{L}$ is given by

$$\frac{d\mathbf{L}}{dt}=\frac{2\tilde{q}^2}{3c^3}\mathbf{r}\times\frac{d^3\mathbf{r}}{dt^3}, \tag{53}$$

which, integrated over the unperturbed Rutherford trajectory, gives the total change of angular momentum in the scattering process,

$$\delta\mathbf{L}=\frac{4\tilde{q}^2}{3c^3}\frac{q_1q_2v_0}{L^2}\left[1-\frac{q_1q_2}{v_0L}\arctan\left(\frac{v_0L}{q_1q_2}\right)\right]\mathbf{L}. \tag{54}$$

At a given scattering angle the angular momentum change is

$$\delta\mathbf{L}=\frac{4}{3}\frac{\tilde{q}^2}{q_1q_2}\left(\frac{v_0}{c}\right)^3\chi(\theta)\mathbf{L}, \tag{55}$$

where

$$\chi(\theta)=\tan^2(\theta/2)\left[1-\frac{\pi-\theta}{2}\tan(\theta/2)\right]. \tag{56}$$

This function is shown in Fig. 3.

The energy loss is readily calculated by differentiating Eq. (30),

$$\frac{\delta E}{E}=\frac{2}{(v_0L)^2}\mathbf{A}\cdot\delta\mathbf{A}-\frac{2}{L^2}\mathbf{L}\cdot\delta\mathbf{L}. \tag{57}$$

If we substitute the expressions for $\delta\mathbf{A}$ and $\delta\mathbf{L}$ in Eqs. (41) and (54), we obtain

$$\frac{\delta E}{E}=\frac{4\tilde{q}^2}{3c^3}\frac{(q_1q_2)^2}{L^3}\times\left\{3\frac{q_1q_2}{v_0L}-\left[1+3\left(\frac{q_1q_2}{v_0L}\right)^2\right]\arctan\left(\frac{v_0L}{q_1q_2}\right)\right\}, \tag{58}$$

or, in terms of the scattering angle,

$$\frac{\delta E}{E}=-\frac{4}{3}\frac{\tilde{q}^2}{q_1q_2}\left(\frac{v_0}{c}\right)^3\xi(\theta), \tag{59}$$

where $\xi(\theta)$ is given in Eq. (51) and shown in Fig. 2.

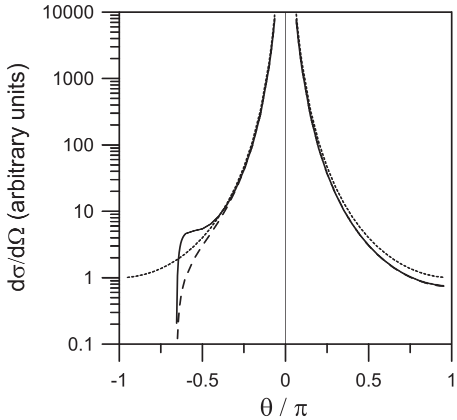

Fig. 4. Rutherford scattering to order c^{-3}. The projectile velocity is $0.4c$, and the target has infinite mass. The two electric charges are of the same magnitude, like (unlike) signs corresponding to positive (negative) scattering angles. The nonrelativistic Rutherford cross section is given by the dotted lines. The dashed lines incorporate c^{-2} corrections, and the solid lines include the c^{-3} radiation damping effects.

V. COMMENTS

Our discussion of radiation damping corrections to Rutherford scattering ignored relativistic effects such as retardation, magnetic forces, and the mass-velocity dependence. These effects give contributions of order c^{-2} to the deflection function and cross section (see Ref. 7), and are generally more important than the c^{-3} radiative corrections. They were not considered here because a c^{-2} correction to the nonrelativistic Rutherford trajectory adds only c^{-5} terms to our perturbative calculation of radiation damping. We can easily write the complete (up to c^{-3}) expansion of the deflection function and scattering cross section by combining the results of Ref. 7 and the present paper. For example, the differential cross section to order c^{-3} is

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_R}{d\Omega}\left[1 - \left(\frac{v_0}{c}\right)^2 h(\theta)\right]\left[1 + 5\frac{\mu}{M}\left(\frac{v_0}{c}\right)^2\right] \times\left[1 + \frac{\tilde{q}^2}{q_1 q_2}\left(\frac{v_0}{c}\right)^3 \xi(\theta)\right], \quad (60)$$

where

$$h(\theta) = \tfrac{1}{2}\tan^2(\theta/2)[1 + (\pi - \theta)\cot\theta] + 1, \quad (61)$$

and $M = m_1 + m_2$. As discussed in Ref. 7, the first correction term accounts for the variation of mass with velocity, and the second term includes magnetic and retardation effects. The last term is the radiative correction calculated in Sec. IV. Note that magnetic and retardation effects simply renormalize the cross section by an angle independent factor.

In Fig. 4 we show the differential cross section for the scattering of a charged particle with $v_0 = 0.4c$ on a fixed target of equal ($\theta > 0$) or opposite ($\theta < 0$) charge. The dotted lines give the nonrelativistic cross section, and the dashed ones show the effect of the c^{-2} relativistic mass correction (retardation and magnetic forces do not show up on a fixed target). The solid lines include the radiation damping effect as given in Eq. (60). We see in Fig. 4 that radiation damping has a very small effect when the charges repel each other. But for an attractive Coulomb force the radiative correction is important (as also seen in Fig. 2), creating a plateau-like structure in the angular distribution. Even though our perturbative results are not reliable for large corrections, such structure is very similar to what is found in exact numerical calculations.[6]

A final point that deserves comment is why our results are not plagued by runaway solutions. The reason is that the Runge–Lenz-based perturbative calculation presented here follows a "reduction of order" approach such as described in Refs. 2 and 10 (see also Ref. 11 for a closely related discussion). This procedure effectively eliminates the additional degrees of freedom introduced in the equations of motion by the time derivative of acceleration, yielding only physically acceptable solutions.

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