# Three methods for calculating the Feynman propagator

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We present three methods for calculating the Feynman propagator for the nonrelativistic harmonic oscillator. The first method was employed by Schwinger a half a century ago, but has rarely been used in nonrelativistic problems since. Also discussed is an algebraic method and a path integral method so that the reader can compare the advantages and disadvantages of each method. 

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## I. INTRODUCTION

The purpose of this paper is to discuss three methods for calculating the Feynman propagator. The methods are applied to the harmonic oscillator so that they can be compared by the reader. The harmonic oscillator was chosen because of its intrinsic interest and because it is the simplest nontrivial system after the free particle (see, for instance, Refs. 1 and 2). The first method we will discuss was developed by Schwinger[3] to treat effective actions in quantum electrodynamics and is based on the solution of the Heisenberg operator equations of motion. The use of proper operator ordering and the subsidiary and initial conditions yields the propagator. The second method is based on algebraic techniques based on factorizing the time evolution operator using the Baker–Campbell–Hausdorff formulas.[4,5] By using factorization, the completeness relations, and the value of the matrix element $\langle x|p\rangle$, we can determine the propagator. This method is close to the one presented in Refs. 6 and 7, but here we will use the Baker–Campbell–Hausdorff formulas in a slightly different way. The third method is a path integral calculation that is based on a recurrence relation for the product of infinitesimal propagators. As far as we know, this recurrence relation has not appeared in previous discussions of the one-dimensional harmonic oscillator path integral, although it is inspired by a similar relation in the three-dimensional system.[8]

To establish our notation, we write the Feynman propagator for a time independent nonrelativistic system with Hamiltonian operator $\hat{H}$ in the form:

$$K(x'',x';\tau)=\theta(\tau)\langle x''|\hat{U}(\tau)|x'\rangle, \tag{1}$$

where $\hat{U}(\tau)$ is the time evolution operator:

$$\hat{U}(\tau)=\exp(-i\hat{H}\tau/\hbar), \tag{2}$$

and $\theta(\tau)$ is the step function defined by

$$\theta(\tau)=\begin{cases}1 & \text{if } \tau\geqslant 0\\ 0 & \text{if } \tau<0.\end{cases} \tag{3}$$

## II. SCHWINGER'S METHOD

This method was introduced in 1951 by Schwinger in the context of relativistic quantum field theory[3] and it has since been employed mainly in relativistic problems, such as the calculation of bosonic[9] and fermionic[10–13] Green's functions in external fields. However, this powerful method is also well suited to nonrelativistic problems, although it has rarely been used in the calculation of nonrelativistic Feynman propagators. Recently, this subject has been discussed in an elegant way by using the quantum action principle.[14] Before this time, only a few papers had used Schwinger's method in this context.[15–17] We adopt here a simpler approach that we think is better suited for students and teachers.

First, observe that for $\tau>0$, Eq. (1) leads to the differential equation for the Feynman propagator:

$$i\hbar\frac{\partial}{\partial\tau}K(x'',x';\tau)=\langle x''|\hat{H}\exp\left(-\frac{i}{\hbar}\hat{H}\tau\right)|x'\rangle. \tag{4}$$

By using the general relation between operators in the Heisenberg and Schrödinger pictures,

$$\hat{O}_H(t)=e^{i\hat{H}t/\hbar}\hat{O}_S e^{-i\hat{H}t/\hbar}, \tag{5}$$

it is not difficult to show that if $|x\rangle$ is an eigenvector of the operator $\hat{X}$ with eigenvalue $x$, then it is also true that

$$\hat{X}(t)\,|x,t\rangle=x|x,t\rangle, \tag{6}$$

where

$$\hat{X}(t)=e^{i\hat{H}t/\hbar}\hat{X}e^{-i\hat{H}t/\hbar}, \tag{7}$$

and $|x,t\rangle$ is defined as

$$|x,t\rangle=e^{i\hat{H}t/\hbar}|x\rangle. \tag{8}$$

Using this notation, the Feynman propagator can be written as:

$$K(x'',x';\tau)=\langle x'',\tau|x',0\rangle, \tag{9}$$

where

$$\hat{X}(\tau)|x'',\tau\rangle=x''|x'',\tau\rangle, \tag{10a}$$

$$\hat{X}(0)|x',0\rangle=x'|x',0\rangle. \tag{10b}$$

The differential equation for the Feynman propagator, Eq. (4), takes the form

$$i\hbar\frac{\partial}{\partial\tau}\langle x'',\tau|x',0\rangle=\langle x'',\tau|\hat{H}|x',0\rangle \quad (\tau>0). \tag{11}$$

The form of Eq. (11) is very suggestive and is the starting point for the very elegant operator method introduced by Schwinger.[3] The main idea is to calculate the matrix element on the right-hand side of Eq. (11) by writing $\hat{H}$ in terms of the operators $\hat{X}(\tau)$ and $\hat{X}(0)$, appropriately ordered. Schwinger's method can be summarized by the following steps:

(1) Solve the Heisenberg equations for the operators $\hat{X}(\tau)$ and $\hat{P}(\tau)$, which are given by:

$$i\hbar \frac{d}{dt}\hat{X}(t)=[\hat{X}(t),\hat{H}], \quad i\hbar \frac{d}{dt}\hat{P}(t)=[\hat{P}(t),\hat{H}]. \tag{12}$$

Equations (12) follow directly from Eq. (5).

(2) Use the solutions obtained in step (1) to rewrite the Hamiltonian operator $\hat{H}$ as a function of the operators $\hat{X}(0)$ and $\hat{X}(\tau)$ ordered in such a way that in each term of $\hat{H}$, the operator $\hat{X}(\tau)$ must appear on the left-hand side, while the operator $\hat{X}(0)$ must appear on the right-hand side. This ordering can be done easily with the help of the commutator $[\hat{X}(0),\hat{X}(\tau)]$ (see Eq. (25)). We shall refer to the Hamiltonian operator written in this way as the ordered Hamiltonian operator $\hat{H}_{\rm ord}(\hat{X}(\tau),\hat{X}(0))$. After this ordering, the matrix element on the right-hand side of Eq. (11) can be readily evaluated:

$$\begin{aligned}\langle x'',\tau|\hat{H}|x',0\rangle &= \langle x'',\tau|\hat{H}_{\rm ord}(\hat{X}(\tau),\hat{X}(0))|x',0\rangle \\ &\equiv H(x'',x';\tau)\langle x'',\tau|x',0\rangle,\end{aligned} \tag{13}$$

where we have defined the function $H$. The latter is a c-number and not an operator. If we substitute this result in Eq. (11) and integrate over $\tau$, we obtain:

$$\langle x'',\tau|x',0\rangle = C(x'',x')\exp\left(-\frac{i}{\hbar}\int^{\tau} H(x'',x';\tau')d\tau'\right), \tag{14}$$

where $C(x'',x')$ is an arbitrary integration constant.

(3) The last step is devoted to the calculation of $C(x'',x')$. Its dependence on $x''$ and $x'$ can be determined by imposing the following conditions:

$$\langle x'',\tau|\hat{P}(\tau)\,|x',0\rangle = -i\hbar\frac{\partial}{\partial x''}\langle x'',\tau|x',0\rangle, \tag{15a}$$

$$\langle x'',\tau|\hat{P}(0)|x',0\rangle = +i\hbar\frac{\partial}{\partial x'}\langle x'',\tau|x',0\rangle. \tag{15b}$$

These equations come from the definitions in Eq. (10) together with the assumption that the usual commutation relations hold at any time:

$$[\hat{X}(\tau),\hat{P}(\tau)]=[\hat{X}(0),\hat{P}(0)]=i\hbar. \tag{16}$$

After using Eq. (15), there is still a multiplicative factor to be determined in $C(x'',x')$. This can be done simply by imposing the propagator initial condition:

$$\lim_{\tau\to 0^+}\langle x'',\tau|x',0\rangle = \delta(x''-x'). \tag{17}$$

Now we are ready to apply this method to a large class of interesting problems. In particular, we shall calculate the Feynman propagator for the harmonic oscillator.

The Hamiltonian operator for the harmonic oscillator can be written as

$$\hat{H}=\frac{\hat{P}^2(\tau)}{2m}+\frac{1}{2}m\omega^2\hat{X}^2(\tau), \tag{18}$$

or

$$\hat{H}=\frac{\hat{P}^2(0)}{2m}+\frac{1}{2}m\omega^2\hat{X}^2(0), \tag{19}$$

because the Hamiltonian operator is time independent, despite the fact that the operators $\hat{P}(\tau)$ and $\hat{X}(\tau)$ are explicitly time dependent. It is matter of choice whether to work with the Hamiltonian operator given by Eq. (18) or by Eq. (19). For simplicity, we choose the latter.

As stated in step (1), we start by writing down the corresponding Heisenberg equations:

$$\frac{d}{dt}\hat{X}(t)=\frac{\hat{P}(t)}{m}, \tag{20a}$$

$$\frac{d}{dt}\hat{P}(t)=-m\omega^2\hat{X}(t), \tag{20b}$$

whose solutions permit us to write for $t=\tau$ that

$$\hat{X}(\tau)=\hat{X}(0)\cos\,\omega\tau+\frac{\hat{P}(0)}{m\omega}\sin\,\omega\tau. \tag{21}$$

For later convenience, we also write the corresponding expression for $\hat{P}(\tau)$:

$$\hat{P}(\tau)=-m\omega\hat{X}(0)\sin\,\omega\tau+\hat{P}(0)\cos\,\omega\tau. \tag{22}$$

To complete step (2) we need to rewrite $\hat{P}(0)$ in terms of $\hat{X}(\tau)$ and $\hat{X}(0)$, which can be done directly from Eq. (21):

$$\hat{P}(0)=\frac{m\omega}{\sin(\omega\tau)}[\hat{X}(\tau)-\hat{X}(0)\cos\,\omega\tau]. \tag{23}$$

If we substitute this result into Eq. (19), we obtain

$$\begin{aligned}\hat{H}=&\frac{m\omega^2}{2\sin^2(\omega\tau)}[\hat{X}^2(\tau)+\hat{X}^2(0)\cos^2(\omega\tau)\\ &-\hat{X}(0)\hat{X}(\tau)\cos(\omega\tau)-\hat{X}(\tau)\hat{X}(0)\cos(\omega\tau)]\\ &+\tfrac{1}{2}m\omega^2\hat{X}^2(0).\end{aligned} \tag{24}$$

Note that the third term in Eq. (24) is not written in the appropriate order. By using the commutation relation

$$\begin{aligned}[\hat{X}(0),\hat{X}(\tau)]&=\left[\hat{X}(0),\hat{X}(0)\cos(\omega\tau)+\frac{\hat{P}(0)}{m\omega}\sin(\omega\tau)\right]\\ &=\frac{i\hbar}{m\omega}\sin(\omega\tau),\end{aligned} \tag{25}$$

it follows immediately that

$$\hat{X}(0)\hat{X}(\tau)=\hat{X}(\tau)\hat{X}(0)+\frac{i\hbar}{m\omega}\sin\,\omega\tau. \tag{26}$$

If we substitute Eq. (26) into Eq. (24), we obtain the ordered Hamiltonian:

$$\hat{H}_{\rm ord}=\frac{m\omega^2}{2\sin^2(\omega\tau)}[\hat{X}^2(\tau)+\hat{X}^2(0)-2\hat{X}(\tau)\hat{X}(0)\cos(\omega\tau)] -\frac{i\hbar\omega}{2}\cot(\omega\tau). \tag{27}$$

Once the Hamiltonian operator is appropriately ordered, we can find the function $H(x'',x';\tau)$ directly from its definition, given by Eq. (13):

$$\begin{aligned}H(x'',x',\tau)&=\frac{\langle x'',\tau|\hat{H}|x',0\rangle}{\langle x'',\tau|x',0\rangle}\\&=\frac{m\omega^2}{2}[(x''^2+x'^2)\csc^2(\omega\tau)\\&\quad-2x''x'\cot(\omega\tau)\csc(\omega\tau)]-\frac{i\hbar\omega}{2}\cot(\omega\tau).\end{aligned} \tag{28}$$

By using Eq. (14), we can express the propagator in the following form:

$$\begin{aligned}\langle x'',\tau|x',0\rangle&=C(x'',x')\exp\left\{-\frac{i}{\hbar}\int^{\tau}d\tau'\left[\frac{m\omega^2}{2}((x''^2+x'^2)\right.\right.\\&\quad\times\csc^2\omega\tau'-2x''x'\cot(\omega\tau)\csc\omega\tau')\\&\quad\left.\left.-\frac{i\hbar\omega}{2}\cot\omega\tau'\right]\right\}.\end{aligned} \tag{29}$$

The integration over $\tau'$ in Eq. (29) can be readily evaluated:

$$\begin{aligned}\langle x'',\tau|x',0\rangle&=\frac{C(x'',x')}{\sqrt{\sin(\omega\tau)}}\exp\left\{\frac{im\omega}{2\hbar\sin(\omega\tau)}\right.\\&\quad\left.\times[(x''^2+x'^2)\cos(\omega\tau)-2x''x']\right\},\end{aligned} \tag{30}$$

where $C(x'',x')$ is an arbitrary integration constant to be determined according to step (3).

The determination of $C(x'',x')$ is done with the aid of Eqs. (15) and (17). However, we need to rewrite the operators $\hat{P}(0)$ and $\hat{P}(\tau)$ in terms of the operators $\hat{X}(\tau)$ and $\hat{X}(0)$, appropriately ordered. For $\hat{P}(0)$ this task has already been done (see Eq. (23)), and for $\hat{P}(\tau)$ we find after substituting Eq. (23) into Eq. (22):

$$\hat{P}(\tau)=m\omega\cot(\omega\tau)[\hat{X}(\tau)-\hat{X}(0)\cos\omega\tau] -m\omega\hat{X}(0)\sin(\omega\tau). \tag{31}$$

Then, by inserting Eqs. (31) and (30) into Eq. (15a) it is not difficult to show that:

$$\frac{\partial C(x'',x')}{\partial x''}=0. \tag{32}$$

Analogously, by substituting Eqs. (23) and (30) into Eq. (15b) we have that $\partial C(x'',x')/\partial x'=0$. The last two relations tell us that $C(x'',x')=C$, that is, it is a constant independent of $x''$ and $x'$. In order to determine the value of $C$, we first take the limit $\tau\to 0^+$ on $\langle x'',\tau|x',0\rangle$. If we use Eq. (30), we find that

$$\begin{aligned}\lim_{\tau\to 0^+}\langle x'',\tau|x',0\rangle&=\lim_{\tau\to 0^+}\frac{C}{\sqrt{\omega\tau}}\exp\left[\frac{im}{2\hbar\tau}(x''-x')^2\right]\\&=C\sqrt{\frac{2\pi i\hbar}{m\omega}}\delta(x''-x').\end{aligned} \tag{33}$$

If we compare this result with the initial condition, Eq. (17), we obtain $C=\sqrt{m\omega/2\pi i\hbar}$. By substituting this result for $C$ into Eq. (30), we obtain the desired Feynman propagator for the harmonic oscillator:

$$\begin{aligned}K(x'',x';\tau)&=\langle x'',\tau|x',0\rangle\\&=\sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega\tau)}}\exp\left\{\frac{im\omega}{2\hbar\sin(\omega\tau)}\right.\\&\quad\left.\times[(x''^2+x'^2)\cos(\omega\tau)-2x''x']\right\}.\end{aligned} \tag{34}$$

In Sec. V we shall see how to extract from Eq. (34) the eigenfunctions and energy eigenvalues for the harmonic oscillator and also how to obtain, starting from the Feynman propagator, the corresponding partition function.

For other applications of this method we suggest the following problems for the interested reader. Calculate the Feynman propagator using Schwinger's method for (i) the constant force problem; (ii) a charged spinless particle in a uniform magnetic field; and (iii) a charged spinless particle in a harmonic oscillator potential placed in a uniform magnetic field.

We finish this section by mentioning that Schwinger's method can be applied to time-dependent Hamiltonians as well.[15,16] It also provides a natural way of establishing the midpoint rule in the path integral formalism (see Sec. IV) when electromagnetic fields are present.[17]

## III. ALGEBRAIC METHOD

The origin of the algebraic method dates back to the beginning of quantum mechanics, with the matrix formulation of Jordan, Heisenberg, and Pauli among others. Here, we present an algebraic method for calculating Feynman propagators which involves manipulations of momentum and position operators.[4,18–21] This is a powerful method because it is connected with the dynamical symmetry groups of the system at hand. A knowledge of the underlying Lie algebra can be used to calculate eigenvalues without explicit knowledge of the eigenfunctions.[7,18,19] It can also be used to calculate propagators for a wide range of problems.[6,22–26] A coherent-state version of the algebraic method for different problems has been discussed also.[23,27] Because the use of these mathematical tools can be a bit cumbersome at first reading, we prefer to explore a simpler version of this method, which is close to that in Ref. 6. For this purpose, the calculation of the propagator for the one-dimensional harmonic oscillator is excellent.

The Hamiltonian operator $\hat{H}$ for a nonrelativistic system can usually be written as a sum of terms involving the operators $\hat{P}$ and $\hat{X}$ which do not commute. Hence, the factorization of the time evolution operator $\hat{U}(\tau)=\exp(-i\tau\hat{H}/\hbar)$ into a product of simpler exponential operators involves some algebra. This algebra deals basically with the commutation relations among these noncommuting operators, and uses formulas generically known as Baker–Campbell–

Hausdorff (see Eq. (35)). The use of those formulas is the essence of the algebraic method, because it is easier to calculate the action of these simpler exponential operators on the states $|x\rangle$ or $|p\rangle$, than to calculate the action on these same states of the original time evolution operator. The algebraic method can be summarized by the following steps:

(1) First rewrite the evolution operator $\hat{U}(\tau)$ as a product of exponentials of the operators $\hat{X}$, $\hat{P}$, and $\hat{P}\hat{X}$. (Note that, in contrast with Schwinger's method, here the operators $\hat{P}$ and $\hat{X}$ are time independent, that is, they are in the Schrödinger representation.) The factorization can be done with the help of the Baker–Campbell–Hausdorff formula[4,5]

$$e^{A}Be^{-A}=C, \tag{35}$$

where $A$, $B$, and $C$ are operators (for simplicity we omit the caret on the operators) and

$$C=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\frac{1}{3!}[A,[A,[A,B]]]+\cdots, \tag{36}$$

valid for any $A$ and $B$. Equation (35) can be iterated as:

$$C^{2}=(e^{A}Be^{-A})(e^{A}Be^{-A})=e^{A}B^{2}e^{-A},$$
$$\vdots$$
$$C^{n}=e^{A}B^{n}e^{-A}. \tag{37}$$

If we expand $\exp(C)$ and identify each power $C^{n}$ in Eq. (37), we find

$$e^{C}=e^{A}e^{B}e^{-A}, \tag{38}$$

which can be inverted to give

$$e^{B}=e^{-A}e^{C}e^{A}. \tag{39}$$

We then identify $B=-i\tau\hat{H}/\hbar$ and find a factorized form of the evolution operator for a conveniently chosen operator $A$. The specific choice for $A$ depends on the explicit form of the Hamiltonian. This factorization can be repeated as many times as needed. Note that, in general, the operator $C$ in Eq. (36), which is an infinite series with $B$ and multiple commutators of $A$ and $B$, is more complicated than the operator $B$ alone, which is proportional to the Hamiltonian. However, if we choose the operator $A$ conveniently, this series can terminate and the remaining terms from the commutators can cancel some of those terms originally present in $B$. A more systematic way of doing this factorization is to use the Lie algebra related to the problem under study. This way will be sketched at the end of this section.

(2) Next substitute the factorized Hamiltonian into the definition of the Feynman propagator $K(x'',x';\tau)$, Eq. (1), and calculate the action of the exponential of the operators $\hat{X}$, $\hat{P}$, and $\hat{P}\hat{X}$ on the state $|x\rangle$. For the operator $\hat{X}$ this calculation is trivial and for $\hat{P}$ we just need to use the closure relation $\mathbb{1}=\int dp|p\rangle\langle p|$ and the matrix element $\langle x|p\rangle=(1/2\pi\hbar)^{1/2}\exp(ixp)$. For the mixed operator $\hat{P}\hat{X}$ we need to use

$$\langle p'|\exp(-i\gamma\hat{P}\hat{X}/\hbar)|p\rangle=e^{-\gamma}\delta(p'-e^{-\gamma}p), \tag{40}$$

where $\gamma$ is an arbitrary parameter to be chosen later. Equation (40) comes from the relation

$$\exp(-i\gamma\hat{P}\hat{X}/\hbar)|p\rangle=e^{-\gamma}|e^{-\gamma}p\rangle. \tag{41}$$

Before we apply the algebraic method to a specific problem, we derive Eq. (41). We first note that

$$\exp(-i\gamma\hat{P}\hat{X}/\hbar)\hat{P}|p\rangle=p\exp(-i\gamma\hat{P}\hat{X}/\hbar)|p\rangle. \tag{42}$$

Equation (42) can be rewritten as

$$[\exp(-i\gamma\hat{P}\hat{X}/\hbar)\hat{P}\ \exp(i\gamma\hat{P}\hat{X}/\hbar)]\exp(-i\gamma\hat{P}\hat{X}/\hbar)|p\rangle$$
$$=p\ \exp(-i\gamma\hat{P}\hat{X}/\hbar)|p\rangle, \tag{43}$$

so that we can use the Baker–Campbell–Hausdorff formulas (35) to rewrite the term in the square brackets as:

$$\exp(-i\gamma\hat{P}\hat{X}/\hbar)\hat{P}\ \exp(i\gamma\hat{P}\hat{X}/\hbar)$$
$$=\left(1+\gamma+\frac{1}{2!}\gamma^{2}+\frac{1}{3!}\gamma^{3}+\ \ldots\right)\hat{P}=e^{\gamma}\hat{P}. \tag{44}$$

If we substitute the above result into Eq. (43), we have

$$\hat{P}[\exp(-i\gamma\hat{P}\hat{X}/\hbar)|p\rangle]=e^{-\gamma}p[\exp(-i\gamma\hat{P}\hat{X}/\hbar)|p\rangle], \tag{45}$$

which shows that $\exp(-i\gamma\hat{P}\hat{X}/\hbar)|p\rangle$ is an eigenstate of the operator $\hat{P}$ with eigenvalue $pe^{-\gamma}$. This eigenstate can be written as $|e^{-\gamma}p\rangle$, up to a constant $C_{\gamma}$, so that

$$\exp(-i\gamma\hat{P}\hat{X}/\hbar)|p\rangle=C_{\gamma}|e^{-\gamma}p\rangle. \tag{46}$$

To determine the constant $C_{\gamma}$, we note that

$$\langle p'|\exp\left(\frac{i}{\hbar}\gamma\hat{X}\hat{P}\right)\exp(-i\gamma\hat{P}\hat{X}/\hbar)|p\rangle=|C_{\gamma}|^{2}\langle e^{-\gamma}p'|e^{-\gamma}p\rangle. \tag{47}$$

If we use the relation $[\hat{X}\hat{P},\hat{P}\hat{X}]=0$, and Eq. (39), we have

$$\langle p'|\exp(i\gamma[\hat{X},\hat{P}]/\hbar)|p\rangle=|C_{\gamma}|^{2}\delta(e^{-\gamma}(p'-p)), \tag{48}$$

so that

$$e^{-\gamma}\ \delta(p-p')=|C_{\gamma}|^{2}\ e^{\gamma}\ \delta(p'-p). \tag{49}$$

Equation (49) determines $C_{\gamma}=e^{-\gamma}$ and finally yields Eq. (41).

Now we are ready to apply the algebraic method to solve some quantum mechanical problems. For the harmonic oscillator the time evolution operator (2) becomes

$$\hat{U}(\tau)=\exp\left[-i\tau\left(\frac{\hat{P}^{2}}{2m}+\frac{1}{2}m\omega^{2}\hat{X}^{2}\right)\Big/\ \hbar\right]. \tag{50}$$

We follow step (1), choose $A=\alpha\hat{X}^{2}$ where $\alpha$ is an arbitrary parameter and $B=-i\tau\hat{H}/\hbar$, and obtain from Eqs. (36) and (38)

$$\exp(\alpha\hat{X}^{2})\exp(-i\tau\hat{H}/\hbar)\exp(-\alpha\hat{X}^{2})$$
$$=\exp\left\{\frac{-i\tau}{\hbar}\left[\frac{\hat{P}^{2}}{2m}+\frac{i\hbar\alpha}{m}(\hat{X}\hat{P}+\hat{P}\hat{X})\right.\right.$$
$$\left.\left.+\frac{m}{2}\left[\omega^{2}-\left(\frac{2\alpha\hbar}{m}\right)^{2}\right]\hat{X}^{2}\right]\right\}. \tag{51}$$

Note that even though the Baker–Campbell–Hausdorff formulas have an infinite number of terms, the number of nonvanishing commutators between $\hat{H}$ and $\hat{X}^{2}$ is finite.

The general idea of the algebraic method is that we want to factorize the time evolution operator. In this case, a step in this direction corresponds to canceling the term with $\hat{X}^2$ on the right-hand side of Eq. (51). This is easily achieved with the choice

$$\alpha=\frac{m\omega}{2\hbar}. \tag{52}$$

Then by using the commutation relation $[\hat{X},\hat{P}]=i\hbar$, we have

$$\exp(-i\tau\hat{H}/\hbar)=e^{i\omega\tau/2}\exp(-\alpha\hat{X}^2)\times\exp\left[-\frac{i}{\hbar}\tau\left(\frac{\hat{P}^2}{2m}+i\omega\hat{P}\hat{X}\right)\right]\exp(\alpha\hat{X}^2). \tag{53}$$

We can repeat step (1) to reduce the above operator containing $\hat{P}^2$ between brackets into a product of simpler terms. This time we need to use products of $\hat{P}^2$ instead of $\hat{X}^2$. If we use Eqs. (35) and (36), we obtain

$$\exp(\beta\hat{P}^2)\left(\frac{\hat{P}^2}{2m}+i\omega\hat{P}\hat{X}\right)\exp(-\beta\hat{P}^2)=\left(\frac{1}{2m}+2\omega\hbar\beta\right)\hat{P}^2+i\omega\hat{P}\hat{X} \tag{54}$$

and to eliminate the term proportional to $\hat{P}^2$ on the right-hand side we take

$$\beta=-\frac{1}{4m\omega\hbar}, \tag{55}$$

which gives from Eqs. (39) and (54)

$$\exp\left[-\frac{i}{\hbar}\tau\left(\frac{\hat{P}^2}{2m}+i\omega\hat{P}\hat{X}\right)\right]=\exp(-\beta\hat{P}^2)\exp\left[-\frac{i}{\hbar}(i\omega\tau)\hat{P}\hat{X}\right]\exp(\beta\hat{P}^2). \tag{56}$$

If we substitute Eq. (56) into Eq. (53), we have that

$$\exp\left(-\frac{i}{\hbar}\tau\hat{H}\right)=e^{i\omega\tau/2}\exp(-\alpha\hat{X}^2)\exp(-\beta\hat{P}^2)\times\exp\left[-\frac{i}{\hbar}(i\omega\tau)\hat{P}\hat{X}\right]\exp(\beta\hat{P}^2)\exp(\alpha\hat{X}^2). \tag{57}$$

Equation (57) is the expression for the time evolution operator written as a product of simpler operators obtained by applying step (1) of the algebraic method.

We next follow step (2), insert Eq. (57) into the definition of the Feynman propagator Eq. (1), and find

$$K(x'',x';\tau)=\exp\left[-\alpha(x''^2-x'^2)+\frac{i\omega\tau}{2}\right]\int\frac{dp\,dp'}{2\pi\hbar}\times\exp\left[\frac{i}{\hbar}(p'x''-px')-\beta(p'^2-p^2)\right]\times\langle p'|\exp\left[-\frac{i}{\hbar}(i\omega\tau)\hat{P}\hat{X}\right]|p\rangle. \tag{58}$$

If we use Eq. (40) with $\gamma=i\omega\tau$, and the definitions (52) and (55), we have

$$K(x'',x';\tau)=\frac{1}{2\pi\hbar}\exp\left[-\frac{m\omega}{\hbar}\left((x''^2-x'^2)+2\frac{(e^{-i\omega\tau}x''-x')^2}{1-e^{-2i\omega\tau}}\right)-\frac{i\omega\tau}{2}\right]\times\int dp\exp\left[-\left(\frac{1-e^{-2i\omega\tau}}{4m\omega\hbar}\right)\times\left(p-2im\omega\frac{(e^{-i\omega\tau}x''-x')}{1-e^{-2i\omega\tau}}\right)^2\right]. \tag{59}$$

This integral has a Gaussian form and can be easily done giving the harmonic oscillator propagator:

$$K(x'',x';\tau)=\sqrt{\frac{m\omega}{2\pi i\hbar\ \sin\omega\tau}}\times\exp\left[\frac{im\omega}{2\hbar\sin\omega\tau}((x''^2+x'^2)\cos\ \omega\tau-2x''x')\right], \tag{60}$$

where we used that $(1-e^{-2i\omega\tau})=2ie^{-i\omega\tau}\sin\omega\tau$ and Euler's formula, $e^{i\omega\tau}=\cos\omega\tau+i\sin\omega\tau$. This result naturally agrees with the one obtained in Sec. III using Schwinger's method.

Before we finish this section, we want to comment that the algebraic method can be discussed on more formal grounds, identifying the underlying Lie algebra, and using it to explicitly solve the problem of interest. For the one-dimensional harmonic oscillator we can find a set of operators

$$L_-=-\tfrac{1}{2}\partial_{xx},\quad L_+=\tfrac{1}{2}x^2,\quad L_3=\tfrac{1}{2}x\partial_x+\tfrac{1}{4}, \tag{61}$$

such that the Hamiltonian operator can be written as $\hat{H}=(\hbar/m)L_-+m\omega^2L_+$. The above operators satisfy the SO(3) Lie algebra

$$[L_+,L_-]=2L_3,\quad [L_3,L_\pm]=\pm L_\pm. \tag{62}$$

This algebra is isomorphic to the usual SU(2) Lie algebra of the angular momenta and can be used to construct specific Baker–Campbell–ausdorff formulas,[20,21] so that given the algebra, the solution arises naturally.[6] If one considers three-dimensional problems, which are more involved because of the presence of terms proportional to $1/r^2$, this algebra can still be used to find the propagator, but the operators have to be generalized.[18,19] These generalized operators can be used to solve a wide range of problems.[22–26]

## IV. PATH INTEGRAL METHOD

The path integral formalism was introduced by Feynman[28] in 1948, following earlier ideas developed by Dirac.[29] In the last few decades, path integral methods have become very popular, mainly in the context of quantum mechanics, statistical physics, and quantum field theory. Since the pioneering textbook of Feynman and Hibbs,[30] many others have been written on this subject, not only in quantum mechanics,[8,31–34] but also in condensed matter,[35] as well as quantum field theory,[36–38] to mention just a few.

In this section we shall apply Feynman's method to again obtain the harmonic oscillator quantum propagator already established in Secs. II and III. The purpose here is to evaluate the corresponding path integral explicitly, without making use of the semiclassical approach which is often used in the literature. Of course this kind of direct calculation already exists in the literature, see for example Refs. 1, 38, and 39. However, we shall present an alternative and very simple procedure.

The path integral expression for the quantum propagator is formally given by

$$K(x_N,x_0;\tau)=\int_{x(0)=x_0 \atop x(\tau)=x_N}[Dx]e^{iS(x)/\hbar}, \tag{63}$$

where $S(x)$ is the action functional:

$$S(x)\equiv\int_{t_0}^{t_N}\left[\frac{1}{2}m\dot{x}^2(t)-V(x(t))\right]dt, \tag{64}$$

and $[Dx]$ is the functional measure. According to Feynman's prescription, we have that:

$$\begin{aligned}K(x_N,x_0;\tau)&=\lim_{N\to\infty \atop \varepsilon\to 0}\sqrt{\frac{m}{2\pi i\hbar\varepsilon}}\int\prod_{j=1}^{N-1}\left(\sqrt{\frac{m}{2\pi i\hbar\varepsilon}}dx_j\right)\\&\quad\times\exp\left\{\frac{i}{\hbar}\sum_{k=1}^{N}\left[\frac{m(x_k-x_{k-1})^2}{2\varepsilon}-\varepsilon V\left(\frac{x_k+x_{k-1}}{2}\right)\right]\right\},\end{aligned} \tag{65}$$

where $N\varepsilon=\tau$. With this prescription, the scenario is the following: summation over all the functions $x$ means to sum over all the polynomials in the plane $(t,x(t))$, starting at $(x_0,t_0)$ and finishing at $(x_N,t_N)$, which gives rise to the integrations over the variables $x_j\equiv x(t_j)$ from $-\infty$ to $\infty$, where $t_j=t_0+j\varepsilon$, with $j=1,2,\ldots,N-1$. Hence, to evaluate a path integral means to calculate an infinite number of ordinary integrals, which requires some kind of recurrence relation.

When electromagnetic potentials are absent as is the case here, it is not necessary to adopt the midpoint rule for the potential $V(x)$ as given by Eq. (65), and other choices can also be made. Instead of using the midpoint rule we shall write the discretized version of the action as

$$S\cong\sum_{j=1}^{N}\frac{m(x_j-x_{j-1})^2}{2\tau_j}-\tau_j\frac{1}{2}(V(x_j)+V(x_{j-1})), \tag{66}$$

where for generality we have taken $\tau_j$ as the $j$th time interval so that $\tau=\Sigma_{j=1}^{N}\tau_j$. Then, we write the Feynman propagator in the form:

$$K(x_N,x_0;\tau)=\lim_{N\to\infty}\int\prod_{j=1}^{N}K(x_j,x_{j-1};\tau_j)\prod_{k=1}^{N-1}dx_k, \tag{67}$$

where the propagator for an infinitesimal time interval is given by

$$\begin{aligned}K(x_j,x_{j-1};\tau_j)=\sqrt{\frac{m}{2\pi i\hbar\tau_j}}\exp\left\{\frac{i}{\hbar}\left[\frac{m(x_j-x_{j-1})^2}{2\tau_j}\right.\right.\\ \left.\left.-\tau_j\frac{1}{2}(V(x_j)+V(x_{j-1}))\right]\right\}.\end{aligned} \tag{68}$$

If we use the Lagrangian for the harmonic oscillator, namely,

$$L(x,\dot{x})=\tfrac{1}{2}m\dot{x}^2-\tfrac{1}{2}m\omega^2x^2, \tag{69}$$

the infinitesimal propagator, given by Eq. (68), takes the form:

$$\begin{aligned}K(x_j,x_{j-1};\tau_j)=\sqrt{\frac{m\omega}{2\pi\hbar}}\sqrt{\frac{1}{\omega\tau_j}}\exp\left\{\frac{im\omega}{2\hbar}\frac{1}{\omega\tau_j}\left[\left(1-\frac{\omega^2\tau_j^2}{2}\right)\right.\right.\\ \left.\left.\times(x_j^2+x_{j-1}^2)\right]-2x_jx_{j-1}\right\}.\end{aligned} \tag{70}$$

To calculate this infinitesimal propagator we now define new variables $\phi_j$ such that:

$$\sin\phi_j=\omega\tau_j, \tag{71}$$

which implies that $\phi_j\cong\omega\tau_j$ and $\cos\phi_j\cong1-\omega^2\tau_j^2/2$. In fact, other variable transformations could be tried, but this is the simplest one that we were able to find that allows an easy iteration through a convolution-like formula. It is also helpful to introduce a function $F$:

$$\begin{aligned}F(\eta,\eta';\phi)=\sqrt{\frac{m\omega}{2\pi i\hbar}}\sqrt{\frac{1}{\sin\phi}}\exp\left[\frac{im\omega}{2\hbar}\frac{1}{\sin\phi}\right.\\ \left.\times(\cos\phi(\eta^2+\eta'^2)-2\eta\eta')\right].\end{aligned} \tag{72}$$

Then, using Eqs. (70)–(72), we can rewrite the harmonic oscillator propagator (67) as:

$$K(x_N,x_0;\tau)=\lim_{N\to\infty}\int\prod_{j=1}^{N}F(x_j,x_{j-1};\phi_j)\prod_{k=1}^{N-1}dx_k. \tag{73}$$

The function $F$ has an interesting property:

$$\int_{-\infty}^{\infty}F(\eta'',\eta;\phi'')F(\eta,\eta';\phi')d\eta=F(\eta'',\eta';\phi''+\phi'). \tag{74}$$

This can be seen by a simple direct calculation. From its definition (72), we have that

$$\begin{aligned}&\int_{-\infty}^{\infty}F(\eta'',\eta;\phi'')F(\eta,\eta';\phi')d\eta\\&=\frac{m\omega}{2\pi i\hbar}\sqrt{\frac{1}{\sin\phi''\sin\phi'}}\exp\left[\frac{im\omega}{2\hbar}\left(\frac{\cos\phi''}{\sin\phi''}\eta''^2\right.\right.\\&\quad\left.\left.+\frac{\cos\phi'}{\sin\phi'}\eta'^2\right)\right]\int_{-\infty}^{\infty}d\eta\exp\left[\frac{im\omega}{2\hbar}(\alpha\eta^2-2\eta\beta)\right],\end{aligned} \tag{75}$$

where we have defined

$$\alpha=\frac{\sin(\phi''+\phi')}{\sin\phi''\sin\phi'},\quad\beta=\frac{\eta''\sin\phi'+\eta'\sin\phi''}{\sin\phi''\sin\phi'}. \tag{76}$$

By completing the square in the integrand of Eq. (75) and calculating the Fresnel integral, we obtain

$$\int_{-\infty}^{\infty} F(\eta'',\eta;\phi'')F(\eta,\eta';\phi')d\eta$$

$$= \frac{m\omega}{2\pi i\hbar}\sqrt{\frac{2\pi i\hbar}{m\omega\alpha}}\sqrt{\frac{1}{\sin\phi''\sin\phi'}}$$

$$\times\exp\left[\frac{im\omega}{2\hbar}\left(\frac{\cos\phi''}{\sin\phi''}\eta''^2+\frac{\cos\phi'}{\sin\phi'}\eta'^2-\frac{\beta^2}{\alpha}\right)\right]. \quad (77)$$

If we use the definitions in Eq. (76), as well as some trivial manipulations with trigonometric functions, it is straightforward to show that

$$\int_{-\infty}^{\infty} F(\eta'',\eta;\phi'')F(\eta,\eta';\phi')d\eta$$

$$= \sqrt{\frac{m\omega}{2\pi i\hbar}}\sqrt{\frac{1}{\sin(\phi''+\phi')}}\exp\left[\frac{im\omega}{2\hbar}\frac{1}{\sin(\phi''+\phi')}\right.$$

$$\left.\times(\cos(\phi''+\phi')(\eta''^2+\eta'^2)-2\eta''\eta')\right], \quad (78)$$

which is precisely Eq. (74). We now define $x''=x_N$, $x'=x_0$, use the fact that $\lim_{N\to\infty}\Sigma_{j=1}^{N}\phi_j=\omega\tau$ and the result in Eq. (73) to finally obtain the desired propagator:

$$K(x'',x';\tau)=F(x'',x';\omega\tau)$$

$$=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega\tau}}$$

$$\times\exp\left[\frac{im\omega}{2\hbar\sin\omega\tau}(\cos\omega\tau(x''^2+x'^2)-2x''x')\right]. \quad (79)$$

This result coincides with the expressions obtained in the previous sections for the harmonic oscillator. Let us now review some of the applications of the Feynman propagator.

## V. EIGENFUNCTIONS, EIGENVALUES AND THE PARTITION FUNCTION

In this section we will show how to obtain the stationary states and the corresponding energy levels, as well as the partition function of the quantum harmonic oscillator, directly from the expression of the Feynman propagator calculated earlier. Although these tasks are well known in the literature (see, for instance, Refs. 38 and 40), we shall present them here for completeness.

To obtain the energy eigenstates and eigenvalues, we need to recast the propagator (79) in a form that permits a direct comparison with the spectral representation for the Feynman propagator given by

$$K(x,x';\tau)=\Theta(\tau)\sum_n \phi_n(x)\phi_n^*(x')e^{-iE_n\tau/\hbar} \quad (\tau>0). \quad (80)$$

If we define the variable $z=e^{-i\omega\tau}$, we can write

$$\sin(\omega\tau)=\frac{1}{2i}\frac{1-z^2}{z}, \quad (81a)$$

$$\cos(\omega\tau)=\frac{1+z^2}{2z}. \quad (81b)$$

We further define $\xi'\equiv\sqrt{m\omega/\hbar}x'$ and $\xi''\equiv\sqrt{m\omega/\hbar}x''$, and express the harmonic oscillator propagator to the form:

$$K(x'',x';\tau)=\sqrt{\frac{m\omega z}{\pi\hbar}}(1-z^2)^{-1/2}$$

$$\times\exp\left\{\frac{1}{1-z^2}\left[2\xi'\xi''z-(\xi'^2+\xi''^2)\left(\frac{1+z^2}{2}\right)\right]\right\} \quad (82)$$

$$=\sqrt{\frac{m\omega z}{\pi\hbar}}(1-z^2)^{-1/2}\exp\left[-\frac{1}{2}(\xi'^2+\xi''^2)\right]$$

$$\times\exp\left[\frac{2\xi'\xi''z-(\xi'^2+\xi''^2)z^2}{1-z^2}\right], \quad (83)$$

where we used the identity

$$\frac{1+z^2}{2(1-z^2)}=\frac{1}{2}+\frac{z^2}{1-z^2}. \quad (84)$$

Now we consider Mehler's formula:[41]

$$(1-z)^{-1/2}\exp\left[\frac{2xyz-(x^2+y^2)z^2}{1-z^2}\right]$$

$$=\sum_{n=0}^{\infty}H_n(x)H_n(y)\frac{z^n}{2^n n!} \quad (|z|<1). \quad (85)$$

However, some care must be taken in order to use Eq. (85) in Eq. (83), because $|z|=1$ and Mehler's formula (85) requires that $|z|<1$. This problem can be circumvented if we add an imaginary part to $\omega$, namely, if we let $\omega\to\omega-i\varepsilon$, and take $\varepsilon\to 0$ after the calculations. Hence, if we use Eq. (85), Eq. (83) takes the form:

$$K(x'',x';\tau)$$

$$=\sqrt{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{2\hbar}(x''^2+x'^2)\right]$$

$$\times\sum_{n=0}^{\infty}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x''\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}x'\right)\frac{e^{-i\omega\tau(n+1/2)}}{2^n n!}, \quad (86)$$

where we have let $x=\xi''=\sqrt{m\omega/\hbar}x''$ and $y=\xi'=\sqrt{m\omega/\hbar}x'$.

If we compare Eq. (86) with the spectral representation (80), we finally obtain the well known results for the energy eigenfunctions (apart from a phase factor) and energy levels, respectively:

$$\phi_n(x)=\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{\pi\hbar}x^2\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \quad (87)$$

$$E_n=(n+\tfrac{1}{2})\hbar\omega. \quad (88)$$

We finish this section by calculating the partition function for the harmonic oscillator. With this purpose in mind, recall that the partition function in general can be written as:

$$Z(\beta)=\mathrm{Tr}\,e^{-\beta\hat{H}}. \quad (89)$$

The trace operation can be taken over a discrete basis, the eigenfunctions of the Hamiltonian itself, or, more conveniently here, over the continuous set of eigenstates of the position operator (denoted by $|x\rangle$):

$$Z(\beta)=\int_{-\infty}^{+\infty} dx\,\langle x|\,e^{-\beta\hat{H}}\,|x\rangle. \tag{90}$$

If we identify the integrand with the Feynman propagator with the end points $x'=x''=x$ and $\beta=i\hbar\tau$ as the imaginary time interval, we have

$$Z(\beta)=\int_{-\infty}^{+\infty} dx\,K(x,x;-i\hbar\beta). \tag{91}$$

Then, from the harmonic oscillator propagator (79), we readily obtain

$$\begin{aligned}K(x,x;-i\hbar\beta)&\\ &=\sqrt{\frac{m\omega}{2\pi\hbar\,\sinh(\omega\beta\hbar)}}\\ &\quad\times\exp\left[-\frac{m\omega}{\hbar\,\sinh(\omega\beta\hbar)}\,(\cosh(\omega\beta\hbar)-1)\,x^2\right],\end{aligned} \tag{92}$$

where we have used $\sin(-i\alpha)=-i\sinh\alpha$ and $\cos(i\alpha)=\cosh\alpha$. By substituting Eq. (92) into Eq. (91), and evaluating the remaining Gaussian integral, we finally obtain

$$\begin{aligned}Z(\beta)&=\sqrt{\frac{m\omega}{2\pi\hbar\sinh(\omega\beta\hbar)}}\\ &\quad\times\int_{-\infty}^{\infty}\exp\left[-\frac{m\omega x^2}{\hbar}\tanh\left(\frac{\omega\beta\hbar}{2}\right)\right]dx\\ &=\frac{1}{2\sinh(\frac{1}{2}\omega\beta\hbar)},\end{aligned} \tag{93}$$

where we used the identities $\cosh\alpha-1=2\sinh^2(\alpha/2)$ and $\sinh(\alpha)=2\sinh(\alpha/2)\cosh(\alpha/2)$. Equation (93) is the partition function for the one-dimensional harmonic oscillator.

## VI. CONCLUSIONS

We have rederived the one-dimensional harmonic oscillator propagator using three different techniques. First we used a method developed by Schwinger that is usually used in quantum field theory, but that is also well suited for nonrelativistic quantum mechanical problems although rarely used. We hope that our presentation of this method will help it become better known among physics teachers and students. Then we presented an algebraic method that deals with the factorization of the time evolution operator using the Baker–Campbell–Hausdorff formulas. We hope that our presentation will motivate teachers and students to learn more about such powerful methods, which are closely connected with the use of Lie algebras. It is worth mentioning that these methods can be applied not only in nonrelativistic quantum mechanics, but also in the context of relativistic theories. Finally, we presented a direct calculation of the Feynman path integral for the one-dimensional harmonic oscillator using a simple but very convenient recurrence relation for the composition of infinitesimal propagators. We hope that the presentation of these three methods together with the calculation of the Feynman propagator will help readers compare the advantages and difficulties of each of them.

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**THE ''RAD'' LAB**

Everyone agreed that the new lab needed to have some sort of title, but a descriptive yet nonrevealing name was hard to find. Finally, one of the Berkeley group suggested calling it the MIT Radiation Laboratory in honor of Lawrence, who was largely responsible for their all being there. The misleading name would account for the large and rather sudden concentration of experimental physicists and cyclotroneers in Cambridge, while at the same time it would be descriptive, in a sly way, of their purpose. In the interests of secrecy, they also hoped the disguise would fool outsiders into thinking that they were engaged in research as altogether remote from the war effort as nuclear fission, which was considered of no practical significance as compared to radar. The ''Rad Lab'' met with unanimous approval and was officially adopted.

Jennett Conant, *Tuxedo Park* (Simon & Schuster, New York, NY, 2002), p. 213.