SPINS Lab 1

1.   Measure the spin projection S_z along the z-axis. This is the experiment that is ready to go when you start the program, as shown below. Each atom is measured to have spin up or spin down, denoted by the arrows and by the $|+\rangle$ and $|-\rangle$ symbols (we will explain the symbols in more detail later) in the figure below. The measured spin projections for these cases are $S_z = \pm\hbar/2$. Run the experiment by selecting Do 1 (ctrl-1) under the Control menu, which sends one atom through the apparatus. Do this repeatedly so you can see the inherent randomness in the measurement process. Try running the experiment continuously (Go) and using the other fixed numbers (10, 100, 1000, 10000).
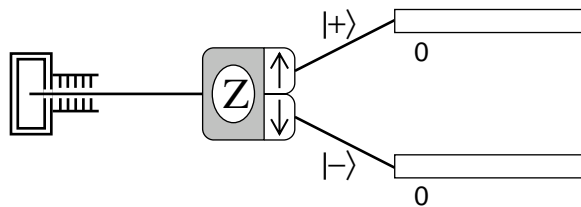
2.   From the above experiments, and from what we have said in class, you will have surmised that the probability for a spin-up measurement is $\mathcal{P} = 1/2$, with the probability for spin down being $(1 - \mathcal{P}) = 1/2$. How can we be certain of this? Let's do a series of experiments and examine the statistics of the data (see appendix for information about statistics). Reset the counters and run the experiment 100 times (ctrl-3). Record the number of counts in the spin-up detector in the table below. Repeat this 10 times to fill up the table (I have already done the 10 atom case). Now put the numbers into your calculator and find the mean $\bar{x}$ and standard deviation s of your data, and the standard deviation of the mean σ_m. Then calculate the experimental estimate of the probability $\mathcal{P}$, its uncertainty $\sigma_\mathcal{P}$, and the relative uncertainty $\sigma_\mathcal{P}/\mathcal{P}$. Do again for 1000 and 10000 atom cases. Are you convinced that $\mathcal{P} = 1/2$? How confident are you?

No. of Atoms (M)	10	100	1000	10000
	7			
	5			
	5			
	8			
Data	5			
(N = 10)	4			
	8			
	2			
	7			
	8			
$\bar{x}$	5.90			
s	2.02			
σ_m	0.64			
$\mathcal{P}$	0.590			
$\sigma_{\mathcal{P}}$	0.064			
$\sigma_{\mathcal{P}}/\mathcal{P}$	0.11			

3.   Now set up an experiment to measure the spin projection S_z along the z-axis twice in succession as shown below. You need an extra analyzer and another counter (see the SPINS notes for help). Run the experiment and note the results. Focus your attention on the second analyzer. The input state is denoted $|+\rangle$ and there are two possible output states $|+\rangle$ and $|-\rangle$. What is the probability that an atom entering the second analyzer (state $|in\rangle = |+\rangle$) exits the spin up port (state $|out\rangle = |+\rangle$) of the second analyzer? This probability is denoted in general as $\mathcal{P}(out) = |\langle out|in\rangle|^2$, and in this case specific case as $\mathcal{P}(+) = |\langle out|in\rangle|^2 = |\langle +|+\rangle|^2$. What is the probability of exiting the spin down port (state $|-\rangle$)? What conclusions can you draw from the measurements performed in this experiment?
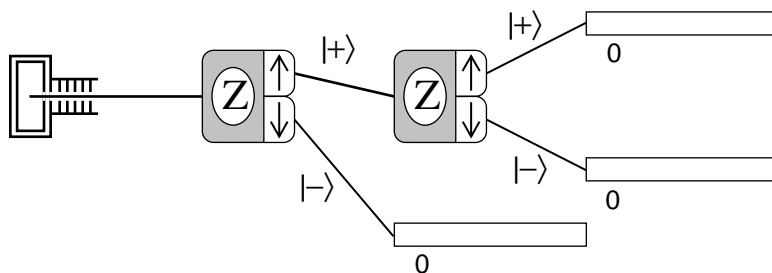
4.　　Using the same apparatus as above (#3), change the orientation directions of the analyzers. You can choose directions *X*, *Y*, or *Z,* which are oriented along the usual *xyz*-axes of a Cartesian coordinate system (ignore the fourth direction $\hat{\mathbf{n}}$ for now). When a direction other than *Z* is chosen, we use a subscript to distinguish the output states (*e.g.*, $|-\rangle_y$). If we allow ourselves to also use the spin down port of the first analyzer as input to the second analyzer (not both up and down at the same time), then there are six possible input states and six possible output states for the second analyzer, which are listed in the table below. Your task is to measure the probabilities $\mathcal{P}(\text{out}) = |\langle \text{out}|\text{in}\rangle|^2$ corresponding to these input and output states. Remember that this is the probability that an atom leaving the first analyzer also makes it through the second analyzer to the appropriate detector, and not the total probability for getting from the oven to the detector. The experiment performed in #3 above (with both analyzers along the *z*-axis) gave the result $|\langle +|+\rangle|^2 = 1$, which is already entered in the table. Now do all other possible combinations and fill in the rest of the table.

$|\langle \text{out}|\text{in}\rangle|^2$	$|+\rangle$	$|-\rangle$	$|+\rangle_x$	$|-\rangle_x$	$|+\rangle_y$	$|-\rangle_y$
$\langle +|$	1					
$\langle -|$						
$_x\langle +|$						
$_x\langle -|$						
$_y\langle +|$						
$_y\langle -|$						

Appendix A: Statistics information

As you see in the experiments, the arrival of an atom at a measurement counter is a random process. We would like to use the results of the experiments to determine the probability $\mathcal{P}$ that governs that random process. In the cases where all the atoms exit one port, then it is clear that the probability is 1 for that output state and zero for the other. However, if we measure 3 spin up atoms and 7 spin down atoms, then we must apply statistical analysis to help us solve the problem. Of course, those results would lead you to conclude that the probability of spin up is $\mathcal{P}(+) = 0.3$ and the probability of spin down is $\mathcal{P}(-) = 0.7$. However, if you performed the experiment a second time and counted 4 spin-up atoms and 6 spin-down atoms, then you would want to revise your estimates.

The questions we thus wish to address are: <u>What is the best estimate of the probability, given the experimental data, and how confident are we of that estimate?</u>

To answer these questions, let's first discuss what results we expect to obtain if we know the probability. Assume that a random process is governed by a probability $\mathcal{P}$, and that each event is independent of all other events. Now assume that we have M of these events and we count the number of successes (*e.g.*, spin-up atoms), which we call n. The probability that we count n spin–up atoms out of M total atoms is determined by the binomial probability distribution, and is given by

$$f_M(n) = \frac{M!}{(M-n)!n!}\mathcal{P}^n(1-\mathcal{P})^{M-n}.$$

This probability distribution is shown in Fig. A1 for the case $M = 10$ and $\mathcal{P} = 0.5$. Thus, for
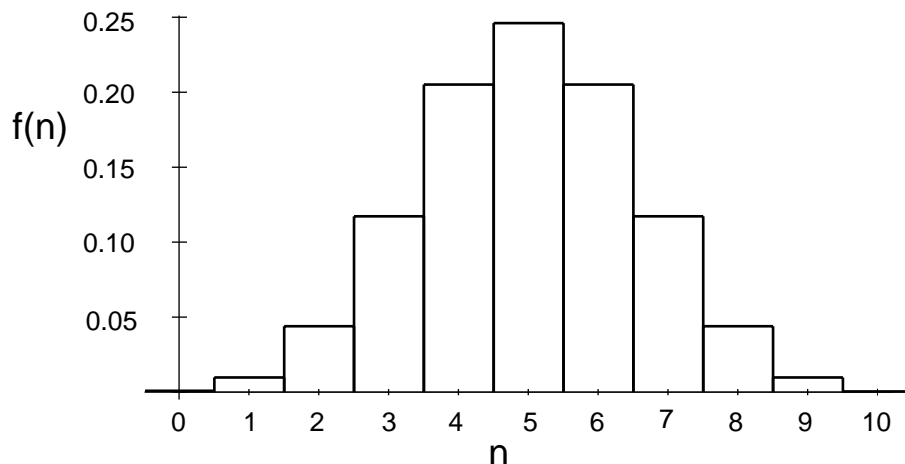

Figure A1. Binomial distribution for 10 events.

example, you expect to count 3 spin-up atoms about 12% of the time ($f_{10}(3) = 0.12$) and 5

spin–up atoms 25% of the time ($f_{10}(5) = 0.25$) in this case. The most obvious conclusion is that

one single measurement of 10 atoms is not too reliable a predictor of the probability $\mathcal{P}$ that an atom

is measured to have spin up.

       To reliably predict the probability we must perform repeated experiments and produce an

experimental histogram of the data akin to the plot in Fig. A1. From the statistical properties of the

histogram we can then estimate the probability and determine an error or uncertainty in that

probability.

       We generally characterize a probability distribution by 2 quantities: (1) the average or mean

or expectation value, which is denoted by $\bar{n}$ or $\langle n \rangle$, and (2) the standard deviation σ, which is the

square root of the variance σ^2. The mean tells you where the distribution is centered and the

standard deviation tells you about the width of the distribution. The mean is obtained as a

weighted average of the possible results:

$$\bar{n} = \sum_n n f(n),$$

where $f(n)$ is the probability of recording n counts. The variance is defined as

$$\sigma^2 = \sum_n (n - \bar{n})^2 f(n).$$

For the binomial distribution, the mean is

$$\bar{n} = M\mathcal{P},$$

and the standard deviation is

$$\sigma = \sqrt{M\mathcal{P}(1 - \mathcal{P})}.$$

       Experimental data is also commonly characterized by these two quantities. Consider an

experiment where a variable x is measured N times to yield a data set x_i. The mean $\bar{x}$ (or average

value) of this data is

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i.$$

The standard deviation s of the data is

$$s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}x_i^2 - \frac{N}{N-1}\bar{x}^2}\,.$$

To connect this firmly to our experiments, assume that the variable x represents the number of times a certain result was obtained in M tries (*e.g.*, M atoms leave the oven and we measure how many end up as spin up). You would thus expect (and it is true) that the best experimental estimates of the parameters $\bar{n}$ and σ of the theoretical distribution are the experimental parameters $\bar{x}$ and s. Thus the experimental estimate of the probability of obtaining the desired result (*e.g.*, the spin-up result) is

$$\mathcal{P} = \frac{\bar{x}}{M}\,.$$

What then is our uncertainty in this estimate? The first guess is to use the standard deviation of the data (divided by M to get a probability) since it is an estimate of the standard deviation of the theoretical probability distribution. However, this is not correct. The standard deviation of the data (and the theoretical probability distribution) tells us how the data are distributed about the mean. The best estimate of the uncertainty of the mean, often called the standard deviation of the mean, is

$$\sigma_m = \frac{s}{\sqrt{N}}\,,$$

which, as you might expect, tells us that we get a better estimate of the mean if we repeat the experiment more times.

A simple example may help to make this all more concrete. Consider an experiment where 10 (M) coins are flipped and the number of heads (x) are counted, and the experiment is repeated 100 times (N). Figure A2 represents data from the experiment. The bars of the histogram tell us how many times a given number of heads occurred. The solid circles (connected by a solid line only as a guide to the eye) are the expected values given that the probability of a head is 1/2; this is just the binomial distribution shown in Fig. A1. The data have a mean of 5.42, with a standard deviation of 1.70, which you can see gives a measure of the width of the distribution of measurements but is much larger than what you might guess is the uncertainty of the mean value. (Note that if we do more experiments (increase N), the standard deviation s will not decrease, but we expect our uncertainty in the mean (*i.e.*, the standard deviation of the mean) to decrease.) From this data we would estimate the probability $\mathcal{P}$ of a head and its uncertainty $\sigma_{\mathcal{P}}$ to be
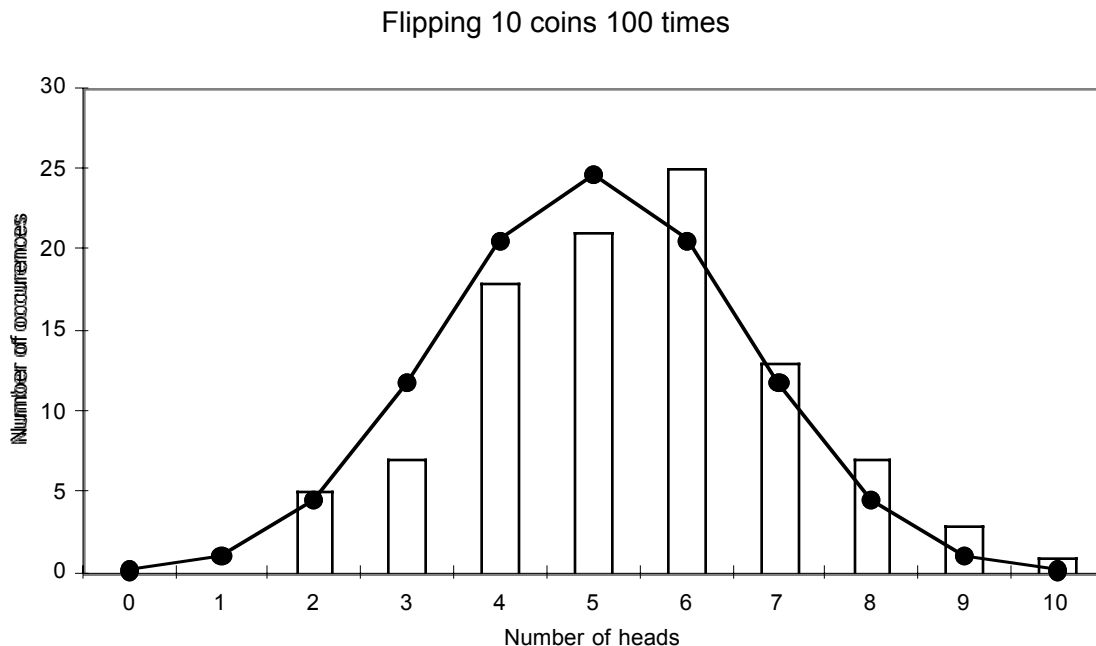
Flipping 10 coins 100 times

Figure A2: Experimental histogram of coin flipping.

$$\mathcal{P} = \frac{\bar{x}}{M} = \frac{5.42}{10} = 0.542$$

$$\sigma_{\mathcal{P}} = \frac{\sigma_m}{M} = \frac{s}{M\sqrt{N}} = \frac{1.70}{10\sqrt{100}} = 0.017$$

Note that the uncertainty is about 3% of the value of the probability. This is a common result in statistics: if you measure something N times, you can generally determine it with a precision of $1/\sqrt{N}$. We already saw this in the standard deviation of the mean. In our counting experiments here, we are actually counting NM atoms and it shouldn't matter whether we measure them as N groups of M or M groups of N, or any other combination; it's all the same data. This is evident if we recall that the standard deviation of the probability distribution scales as $\sqrt{M}$. Thus we expect the uncertainty in the probability to scale like:

$$\sigma_{\mathcal{P}} = \frac{\sigma_m}{M} = \frac{s}{M\sqrt{N}} \propto \frac{\sqrt{M}}{M\sqrt{N}} = \frac{1}{\sqrt{MN}}.$$

In the coin tossing example above $NM = 1000$ flips, so $1/\sqrt{1000} \approx 3\%$. In the 10 atoms case shown in #2 of the lab above, $NM = 100$ atoms, so $1/\sqrt{100} = 10\%$.

Note that the experimental estimate of the probability in the coin tossing example above differs from what we know the real value to be by about 2.5 times the standard deviation. This is only expected to happen 1.5% of the time, but it can happen. We expect our results to be within one standard deviation 68% of the time and within 2 standard deviations 95% of the time.